

Lecture 12

Recall the Non-Rectangular Geometry: 1. Use Cartesian grid: modify discrete operators near boundaries; 2. Mesh that fits boundary. The challenge is to find the map $\bar{x} = (x, y) \leftrightarrow \bar{\xi} = (\xi, \eta)$. Consider the map $\bar{x} = F(\bar{\xi})$. Let $(J_{ij}) = \left(\frac{\partial F_i}{\partial \xi_j}\right)$ be the Jacobian matrix and $g = J^T J$ be the metric tensor. Then, we have

$$\Delta u = \sum_i \sum_j \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial \xi_i} \left(\sqrt{|g|} g_{ij}^{-1} \frac{\partial u(\bar{\xi})}{\partial \xi_j} \right).$$

Now, we begin to discuss the *Hyperbolic* equation.

The first order system $u_t + Au_x = 0$ is hyperbolic if A has real eigenvalues and is diagonalizable. The equation $u_t + (F(u))_x = 0$ is hyperbolic if the Jacobian of F has real eigenvalues and is diagonalizable. In the more spatial dimensions, it is written as $u_t + \nabla \cdot (F(u)) = 0$.

First consider a simple scalar equation

$$u_t + au_x = 0$$

on the real line with initial data $u(x, 0) = u_0(x)$. The solution is

$$u(x, t) = u_0(x - at),$$

i.e., the initial data translates at speed a (no smoothing). In the past, we show that Forward Euler in time, centered difference in space gave an unstable scheme:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a \frac{u_{j+1}^n - u_{j-1}^n}{2h} = 0.$$

For Von-Neumann analysis, we have

$$g(\xi) = 1 - \frac{a\Delta t}{h} \sin(\xi h) i \Rightarrow |g(\xi)| \geq 1.$$

i.e., the numerical scheme is growing. Note $|g(\xi)|^2 = 1 + \left(\frac{a\Delta t}{h}\right)^2 \sin^2(\xi h)$, which we can make it stable if $\Delta t \rightarrow 0$ faster than $h \rightarrow 0$. For example, let $\Delta t = ah^2$, then

$$|g(\xi)|^2 = 1 + \Delta t \left(\frac{a^2 \Delta t}{h^2}\right) \sin^2(\xi h) = 1 + a\Delta t \sin^2(\xi h) = 1 + \alpha \Delta t.$$

So $\Delta t = O(h^2)$ gives stability (not strong stability). But this restriction is too tight. There are other explicit schemes for which $\Delta t = O(h)$ for stability. We could use centered in space but different time stepping, thus needing the imaginary axis in the region of absolute stability. Some Adams methods, 3 or 4 step RK will work.

Write FECS as

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2h} (u_{j+1}^n - u_{j-1}^n).$$

Replacing u_j^n with a spatial average $\frac{1}{2}(u_{j+1}^n + u_{j-1}^n)$ (Lax-Friedrichs Scheme) yields

$$u_j^{n+1} = \frac{1}{2} (u_{j+1}^n + u_{j-1}^n) - \frac{a\Delta t}{2h} (u_{j+1}^n - u_{j-1}^n).$$

Let $\nu = \frac{a\Delta t}{h}$ be the courant #, we get

$$\begin{aligned} u_j^{n+1} &= \frac{1}{2} (u_{j+1}^n + u_{j-1}^n) - \frac{\nu}{2} (u_{j+1}^n - u_{j-1}^n), \\ g(\xi) &= \frac{1}{2} (e^{i\xi h} + e^{-i\xi h}) - \frac{\nu}{2} (e^{i\xi h} - e^{-i\xi h}) = \cos(\xi h) - \nu i \sin(\xi h). \end{aligned}$$

Thus,

$$|g(\xi)|^2 = \cos^2(\xi h) + \nu^2 \sin^2(\xi h) \leq 1 \text{ if } \nu^2 \leq 1,$$

i.e., $\nu^2 \leq 1$ leads to stability. Note $\nu^2 \leq 1 \iff |\nu| \leq 1 \iff \left| \frac{a\Delta t}{h} \right| \leq 1 \iff |\Delta t| \leq \frac{h}{|a|}$, i.e., $\Delta t = O(h)$.

As we all know, in order to get convergence, we also need consistency. It is stable but is it consistent? If yes, how accurate? Note

$$\begin{aligned} \frac{u_j^{n+1} - u_j^n}{\Delta t} &= \frac{1}{2\Delta t} (u_{j+1}^n - 2u_j^n + u_{j-1}^n) - \frac{a}{2h} (u_{j+1}^n - u_{j-1}^n) \Rightarrow \\ u_t + O(\Delta t) &= \frac{h^2}{2\Delta t} u_{xx} + O(h^4/\Delta t) - au_x + O(h^2) \Rightarrow \\ u_t + au_x &= O(\Delta t) + O(h^2) + O(h^2/\Delta t), \end{aligned}$$

which is consistent as long as $h^2/\Delta t \rightarrow 0$ as $\Delta t, h \rightarrow 0$. This also gives a clue about how we obtain stability. Note

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a \frac{u_{j+1}^n - u_{j-1}^n}{2h} = \epsilon \left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} \right),$$

where $\epsilon = \frac{h^2}{2\Delta t}$. From HW1, we have the stability if $\nu^2 \leq 2\mu \leq 1$, where $\mu = \frac{\epsilon\Delta t}{h^2}$. In Lax-Friedrichs, $2\mu = \frac{2\epsilon\Delta t}{h^2} = 2 \frac{h^2}{2\Delta t} \frac{\Delta t}{h^2} = 1$, i.e., $\nu^2 \leq 1$, so no additional stability constraint from the ‘artificial’ diffusion.