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CN and BE are both A -stable. BE is L -stable but CN is not. To study the stability of the numerical scheme, let us recall the study of the eigenvalue problem $u' = \lambda u$. From those numerical schemes, we obtained the recursive relation $u^{n+1} = R(z)u^n$ here $z = \lambda\Delta t$. For BE scheme, we have $R(z) = \frac{1}{1-z}$ and for CN scheme, we have $R(z) = \frac{1+z/2}{1-z/2}$. The stability region of diffusion equations consists of points z so that λ is negative and real for each z . To understand how large time steps effect the numerical solutions, we observe the behavior of the numerical solutions when $z \rightarrow -\infty$ with z real. For BE scheme, we have $R(z) \rightarrow 0$ and for CN scheme, we have $R(z) \rightarrow -1$ (For diffusion equations, high spacial frequencies oscillate in time and decay very slowly).

Definition 0.1. A numerical scheme is called A -stable if $|R(z)| < 1$ for $Re(R(z)) < 0$ and is called L -stable if $|R(z)| \rightarrow 0$ as $|z| \rightarrow \infty$.

For example, BDF methods are L -stable and BDF 1 and BDF 2 are A -stable. Recall the BDF 2 method:

$$3u^{n+1} - 4u^n + u^{n-1} = 2\Delta t f(u^{n+1}),$$

which is LA -stable and has second order accuracy. Let us consider another numerical method, TR-BDF 2¹

$$\begin{aligned} u^* &= u^n + \frac{\Delta t}{4}(f(u^n) + f(u^*)) \\ u^{n+1} &= \frac{1}{3}(4u^* - u^n + \Delta t f(u^{n+1})). \end{aligned}$$

0.1 Fractional Stepping/Time Splitting

Consider

$$u_t = A(u) + B(u),$$

where A and B are differential or algebraic operators. We solve $u_t = A(u)$ with initial data $u(t_n)$ and solve for time length Δt to get u^* ; then solve $u_t = B(u)$ with initial data u^* and solve for time length Δt to get u^{n+1} . Is it true that $u^{n+1} \approx u(t_{n+1})$? We can analyze the error of the split separately from the numerical error. Let us consider the linear case

$$u_t = Au + Bu$$

with initial data $u^n = u(t_n)$. We know that the exact solution for this equation is

$$u(t_{n+1}) = e^{(A+B)\Delta t} u(t_n).$$

If we use splitting method, we know $u^* = e^{A\Delta t} u^n$ and $u^{n+1} = e^{B\Delta t} e^{A\Delta t} u^n$. Note that in general

$$e^{B\Delta t} e^{A\Delta t} \neq e^{(A+B)\Delta t}.$$

To compute the splitting error,

$$u(t_{n+1}) - u^{n+1} = (e^{(A+B)\Delta t} - e^{B\Delta t} e^{A\Delta t}) u^n.$$

¹the implicit R-K method

We perform the Taylor expansion of the above equation as $\Delta t \rightarrow 0$:

$$e^{(A+B)\Delta t} = I + \Delta t(A + B) + \frac{\Delta t^2}{2}(A + B)^2 + \mathcal{O}(\Delta t^3)$$

and

$$e^{B\Delta t}e^{A\Delta t} = I + \Delta t(A + B) + \frac{\Delta t^2}{2}(A^2 + 2BA + B^2) + \mathcal{O}(\Delta t^3).$$

Then we obtained

$$e^{(A+B)\Delta t} - e^{B\Delta t}e^{A\Delta t} = \frac{\Delta t^2}{2}[A, B] + \mathcal{O}(\Delta t^3),$$

where $[A, B] = AB - BA$. Therefore the splitting error is $u(t_{n+1}) = u^{n+1} + \mathcal{O}(\Delta t^2)$. A single step error of the splitting method is $\mathcal{O}(\Delta t^2)$ and thus it is a first order method. How to get a second order splitting errors? Let us introduce the Strang splitting: we take a half time step to solve $u_t = A(u)$, and a full time step to solve $u_t = B(u)$, and a half time step to solve $u_t = A(u)$. The errors can be obtained:

$$e^{A\frac{\Delta t}{2}}e^{B\Delta t}e^{A\frac{\Delta t}{2}} = e^{(A+B)\Delta t} + \mathcal{O}(\Delta t^3)$$

which gives a second order time stepping scheme. Here is another way of splitting

$$\begin{aligned} u^n &\xrightarrow[\Delta t]{A} u^* \xrightarrow[\Delta t]{B} u^{n+1} \\ u^{n+1} &\xrightarrow[\Delta t]{B} u^{**} \xrightarrow[\Delta t]{A} u^{n+2}, \end{aligned}$$

which looks like Strang splitting every other step.

Remark. Here is a nonsplit scheme. Suppose A is stiff and B is nonstiff. We use CN for A and AB2 for B :

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{1}{2}(Au^{n+1} + Au^n) + \frac{3}{2}Bu^n - \frac{1}{2}Bu^{n-1}.$$

Let us recall the LOD for diffusion equation

$$\begin{aligned} \left(I - \frac{b\Delta t}{2}L_x\right)u^* &= \left(I + \frac{b\Delta t}{2}L_y\right)u^n \\ \left(I - \frac{b\Delta t}{2}L_y\right)u^{n+1} &= \left(I + \frac{b\Delta t}{2}L_x\right)u^*, \end{aligned}$$

which looks like first order accurate if L_x and L_y do not commute.

$$u_t = \frac{1}{2}(Lu^{n+1} + Lu^n) = \frac{1}{2}(L_x(u^{n+1} - u^n) + L_y(u^{n+1} + u^n)).$$

Note that ∂_x^2 and ∂_y^2 do commute but L_x and L_y may not. L_x and L_y commute for periodic boundaries and constant coefficients. In this case, there is no splitting error. For other boundaries, choose boundary conditions for u^* to push the splitting error to higher order. Variable coefficient always get a first order split.