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CN and BE are both A-stable. BE is L-stable but CN is not. To study the stability of the numerical scheme, we apply the scheme to the linear problem $u' = \lambda u$ to obtain the recursion relation $u^{n+1} = R(z)u^n$, where $z = \lambda \Delta t$.

$$R(z) = \frac{1}{1-z} \quad \text{for backward Euler}$$
$$R(z) = \frac{1+z/2}{1-z/2} \quad \text{for Crank-Nicolson}$$

For the diffusion equation, λ is negative and real. To understand how large time steps affect the numerical solutions, we observe the behavior of the numerical solutions when $z \to -\infty$ with z real. For BE scheme, we have $R(z) \to 0$ and for CN scheme, we have $R(z) \to -1$. For the diffusion equation the largest eigenvalues correspond to highest frequency modes. Thus for $R(z)$ close to -1, we will see the high spatial frequency components of the solution oscillate in time and decay very slowly. For these types of problems A-stability not enough to ensure good behavior of the numerical method. We need a different kind of stability, L-stability.

Definition 0.1. A numerical scheme is called A-stable if $|R(z)| < 1$ for $Re\,(R(z)) < 0$ and is called L-stable if $|R(z)| \to 0$ as $|z| \to \infty$.

For example, BDF methods are L-stable; BDF 1 (Backward-Euler) and BDF 2 are A-stable. Recall the BDF 2 method:

$$3u^{n+1} - 4u^n + u^{n-1} = 2\Delta t f(u^{n+1}),$$

which is L and A-stable and is second order accurate. Let us consider another numerical method, TR-BDF 2[1]

$$\begin{aligned} u^* &= u^n + \frac{\Delta t}{4}(f(u^n) + f(u^*)) \\ u^{n+1} &= \frac{1}{3}(4u^* - u^n + \Delta t f(u^{n+1})). \end{aligned}$$

Fractional Stepping/Time Splitting

Consider

$$u_t = A(u) + B(u),$$

where A and B are differential or algebraic operators. In the simplest fractional step method, we solve $u_t = A(u)$ with initial data $u(t_n)$ and solve for time length Δt to get u^*; then solve $u_t = B(u)$ with initial data u^* and solve for time length Δt to get u^{n+1}. Is it true that $u^{n+1} \approx u(t_{n+1})$? We can analyze the error of the split separately from the numerical error. Let us consider the linear case

$$u_t = Au + Bu$$

[1] an implicit R-K method

with initial data $u^n = u(t_n)$. We know that the exact solution for this equation is

$$u(t_{n+1}) = e^{(A+B)\Delta t} u(t_n).$$

If we use the splitting method described above, $u^* = e^{A\Delta t} u^n$ and $u^{n+1} = e^{B\Delta t} e^{A\Delta t} u^n$. Note that in general

$$e^{B\Delta t} e^{A\Delta t} \neq e^{(A+B)\Delta t}.$$

To compute the splitting error,

$$u(t_{n+1}) - u^{n+1} = \left(e^{(A+B)\Delta t} - e^{B\Delta t} e^{A\Delta t}\right) u^n.$$

We perform the Taylor expansion of the above equation as $\Delta t \to 0$:

$$e^{(A+B)\Delta t} = I + \Delta t(A+B) + \frac{\Delta t^2}{2}(A+B)^2 + \mathcal{O}(\Delta t^3),$$

and

$$e^{B\Delta t} e^{A\Delta t} = I + \Delta t(A+B) + \frac{\Delta t^2}{2}(A^2 + 2BA + B^2) + \mathcal{O}(\Delta t^3).$$

Subtracting these two expansions,

$$e^{(A+B)\Delta t} - e^{B\Delta t} e^{A\Delta t} = \frac{\Delta t^2}{2}[A, B] + \mathcal{O}(\Delta t^3),$$

where $[A, B] = AB - BA$. Therefore the splitting error is $u(t_{n+1}) = u^{n+1} + \mathcal{O}(\Delta t^2)$. A single step error of the splitting method is $\mathcal{O}(\Delta t^2)$ and thus it is a first order method. How to get a second order splitting errors? Let us introduce the Strang splitting: we take a half time step to solve $u_t = A(u)$, and a full time step to solve $u_t = B(u)$, and a half time step to solve $u_t = A(u)$. The errors can be obtained:

$$e^{A\frac{\Delta t}{2}} e^{B\Delta t} e^{A\frac{\Delta t}{2}} = e^{(A+B)\Delta t} + \mathcal{O}(\Delta t^3)$$

which gives a second order time stepping scheme. Here is another way of splitting

$$\begin{array}{ccccc} u^n & \xrightarrow[\Delta t]{A} & u^* & \xrightarrow[\Delta t]{B} & u^{n+1} \\ u^{n+1} & \xrightarrow[\Delta t]{B} & u^{**} & \xrightarrow[\Delta t]{A} & u^{n+2}, \end{array}$$

which looks like Strang splitting every other step.

Remark. Here is a nonsplit, second-order accurate scheme. Suppose A is stiff and B is nonstiff. We use CN for A and AB2 for B:

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{1}{2}(Au^{n+1} + Au^n) + \frac{3}{2}Bu^n - \frac{1}{2}Bu^{n-1}.$$

Recall the LOD method for the diffusion equation,

$$\begin{aligned} \left(I - \frac{b\Delta t}{2} L_x\right) u^* &= \left(I + \frac{b\Delta t}{2} L_x\right) u^n \\ \left(I - \frac{b\Delta t}{2} L_y\right) u^{n+1} &= \left(I + \frac{b\Delta t}{2} L_y\right) u^*, \end{aligned}$$

which looks like first order accurate if L_x and L_y do not commute. Note that ∂_x^2 and ∂_y^2 do commute but L_x and L_y may not. L_x and L_y commute for periodic boundaries and constant coefficients. In this case, there is no splitting error. For other boundaries, we could choose boundary conditions for u^* to push the splitting error to higher order. Variable coefficient always give first-order accuracy for LOD.