Math 228B

Feb.12, 2009

Recall approaches to numerically solving PDEs in non-rectangular geometry:

- Use Cartesian grid and modify discrete operators near boundaries
- Use a mesh that fits the boundary.

The challenge is to find the map that transforms the domain to a rectangle (or something easy to discretize): $\bar{x} = (x, y) \leftrightarrow \bar{\xi} = (\xi, \eta)$. Once you have the map, you must transform the PDE. Consider the map $\bar{x} = F\left(\bar{\xi}\right)$. Let $= \left(\frac{\partial F_i}{\partial \xi_j}\right)$ be the Jacobian matrix and $g = J^T J$ be the metric tensor. Then, the Laplacian in the $\bar{\xi}$ coordinates is

$$\Delta u = \sum\nolimits_i \sum\nolimits_j \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial \xi_i} \left(\sqrt{|g|} g_{ij}^{-1} \frac{\partial u\left(\bar{\xi}\right)}{\partial \xi_j} \right).$$

Note that in general, we must solve a variable coefficient problem in the transformed coordinates. The coefficients are related to the transformation.

Hyperbolic equations

The first order system $u_t + A u_x = 0$ is hyperbolic if A has real eigenvalues and is diagonalizable. The equation $u_t + (F(u))_x = 0$ is hyperbolic if the Jacobian of F has real eigenvalues and is diagonalizable. In the more spatial dimensions, this equation is of the form $u_t + \nabla \cdot (F(u)) = 0$.

First consider a simple scalar equation

$$u_t + a u_x = 0$$

on the real line with initial data $u(x, 0) = u_0(x)$. The solution is

$$u(x, t) = u_0(x - at),$$

i.e., the initial data translates at speed a(no smoothing). In the past, we show that Forward Euler in time, centered difference in space gave an unstable scheme:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a \frac{u_{j+1}^n - u_{j-1}^n}{2h} = 0.$$

From Von-Neumann analysis, we have

$$g(\xi) = 1 - \frac{a \Delta t}{h} \sin(\xi h)\, i \Rightarrow |g(\xi)| \geq 1.$$

i.e., the numerical solution is growing, but the solution to the PDE does not grow. Note $|g(\xi)|^2 = 1 + \left(\frac{a\Delta t}{h}\right)^2 \sin^2(\xi h)$, which we can make it stable if $\Delta t \to 0$ faster than $h \to 0$. For example, let $\Delta t = ah^2$, then

$$|g(\xi)|^2 = 1 + \Delta t \left(\frac{a^2 \Delta t}{h^2}\right) \sin^2(\xi h) = 1 + a\Delta t \sin^2(\xi h) = 1 + \alpha \Delta t.$$

So $\Delta t = O(h^2)$ gives stability (not strong stability). But this restriction is too tight. There are other explicit schemes for which $\Delta t = O(h)$ for stability. We could use centered in space with a different time stepping scheme. We need to use a scheme that includes some of the imaginary axis in the region of absolute stability. Some Adams methods methods have this property, as do some 3 or 4 step RK methods.

How to get a stable scheme? Write FECS as

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2h}\left(u_{j+1}^n - u_{j-1}^n\right).$$

Replacing u_j^n with a spatial average $\frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right)$ yields the Lax-Friedrichs Scheme

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{a\Delta t}{2h}\left(u_{j+1}^n - u_{j-1}^n\right).$$

Let $\nu = \frac{a\Delta t}{h}$ be the Courant number. The amplification factor is

$$g(\xi) = \frac{1}{2}\left(e^{i\xi h} + e^{-i\xi h}\right) - \frac{\nu}{2}\left(e^{i\xi h} - e^{-i\xi h}\right) = \cos(\xi h) - \nu i \sin(\xi h).$$

Thus,

$$|g(\xi)|^2 = \cos^2(\xi h) + \nu^2 \sin^2(\xi h) \le 1 \text{ if } \nu^2 \le 1,$$

i.e., $\nu^2 \le 1$ leads to stability. Note $\nu^2 \le 1 \iff |\nu| \le 1 \Leftrightarrow \left|\frac{a\Delta t}{h}\right| \le 1 \Leftrightarrow |\Delta t| \le \frac{h}{|a|}$, i.e., $\Delta t = O(h)$.

In order to get convergence, we also need consistency. It is stable but is it consistent? If yes, how accurate? Note

$$\begin{aligned}
\frac{u_j^{n+1} - u_j^n}{\Delta t} &= \frac{1}{2\Delta t}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) - \frac{a}{2h}\left(u_{j+1}^n - u_{j-1}^n\right) \Rightarrow \\
u_t + O(\Delta t) &= \frac{h^2}{2\Delta t} u_{xx} + O\left(h^4/\Delta t\right) - a u_x + O\left(h^2\right) \Rightarrow \\
u_t + a u_x &= O(\Delta t) + O\left(h^2\right) + O\left(h^2/\Delta t\right),
\end{aligned}$$

which is consistent as long as $h^2/\Delta t \to 0$ as $\Delta t, h \to 0$. This also gives a clue about how we obtain stability. Lax-Friedrichs can be written as

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2h} = \epsilon\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}\right),$$

where $\epsilon = \frac{h^2}{2\Delta t}$. From HW1, we have the stability if $\nu^2 \leq 2\mu \leq 1$, where $\mu = \frac{\epsilon\Delta t}{h^2}$. In Lax-Friedrichs, $2\mu = 2\frac{\epsilon\Delta t}{h^2} = 2\frac{h^2}{2\Delta t}\frac{\Delta t}{h^2} = 1$, i.e., $\nu^2 \leq 1$, so no additional stability constraint from the 'artificial' diffusion.