MAT 228B

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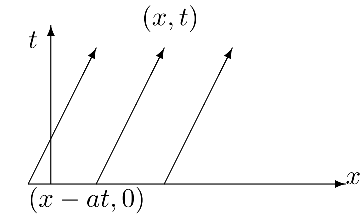
Recall the heat equation $u_t + au_x = 0$. The Lax-Friedrichs gives stability restriction $|\frac{a\Delta t}{h}| \leq 1$. This is a common constraint for explicit methods.

CFL condition

In hyperbolic equations, there is a finite speed of propagation. Let's consider this when designing numerical schemes.

The domain of dependence of the point (x, t) is the set of all points that the solution depends on at (x, t).

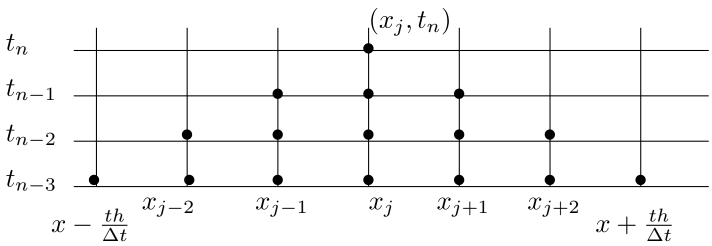


For $u_t + au_x = 0$, the solution is constant along the characteristics $x - at = x_0$.

The domain of dependent(DOD for short) is the single point $(x - at, 0)$

What is DOD of the point (x, t) for $u_t = u_{xx}$? The DOD is $\mathbb{R}$.

For a numerical method, we have the idea of the numerical domain of dependence. The points in discrete space-time on which the discrete solution at (x_j, t_n) depends are



assume it is a three-point centered explicit scheme

The bullets represent the DOD of (x_j, t_n)back in each time step, starting from t_n

What if we refine $r = \frac{\Delta t}{h}$ constant?

Numerical DOD is constrained in the interval $[x - \frac{t}{r}, x + \frac{t}{r}]$, as $h, \Delta t \to 0$, with $r = \frac{\Delta t}{h}$ fixed

It seems that for the solution of the numerical scheme to converge to the solution of the PDE, we should have that $x - at \in [x - \frac{t}{r}, x + \frac{t}{r}]$.

$$x - \frac{t}{r} \leq x - at \leq x + \frac{t}{r}$$

$$-\frac{1}{r} \leq a \leq \frac{1}{r}$$

$$|ar| \leq 1$$

$$|\frac{a\Delta t}{h}| \leq 1$$

This is a necessary condition for convergence

The CFL condition

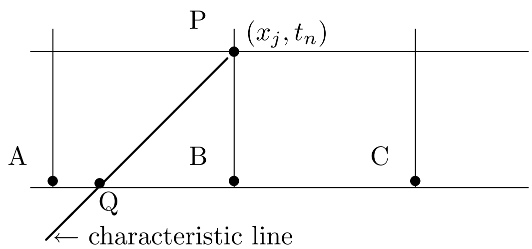
The domain of dependence of the PDE must be contained in the domain of dependence of the numerical scheme. (Necessary condition for convergence)

CFL is necessary but not sufficient: e.g. FE centered space is not stable.

The solution of the PDE motivates using a one-sided difference.

Upwind Method

Assume $a > 0$



compute $u_j^n, p = (x_j, t_n).u(P) = u(Q)$

How to approximate $u(Q)$?

We can interpolate using $u(A)$ and $u(B)$

$$A = (x_{j-1}, t_{n-1}), B = (x_j, t_{n-1}), Q = (x_j - a\Delta t, t_{n-1})$$

$$\begin{aligned}\overrightarrow{QA} &= x_j - a\Delta t - x_{j-1} \\ &= h - a\Delta t \\ &= h(1-\nu)\end{aligned}$$

$$\overrightarrow{QB} = a\Delta t = h\nu$$

$$u_j^n = u(P) = u(Q)$$

$$\begin{aligned}u(Q) &= \frac{\overrightarrow{QB}}{\overrightarrow{AB}} u(A) + \frac{\overrightarrow{QA}}{\overrightarrow{AB}} u(B) \\ &= \nu u(A) + (1-\nu) u(B)\end{aligned}$$

$$\begin{aligned}u_j^n &= \nu u_{j-1}^{n-1} + (1-\nu) u_j^{n-1} \\ &= u_j^{n-1} - \nu(u_j^{n-1} - u_{j-1}^{n-1}) \\ &= u_j^{n-1} - \frac{a\Delta t}{h}(u_j^{n-1} - u_{j-1}^{n-1})\end{aligned}$$

$$with \nu = \frac{a\Delta t}{h}$$

$$\frac{u_j^n - u_j^{(n-1)}}{\Delta t} + \frac{a}{h}(u_j^{n-1} - u_{j-1}^{n-1}) = 0$$

This is forward in time, backward in space.
The Upwind scheme is

$$u_j^{n+1} = \begin{cases} u_j^n - \frac{a\Delta t}{h}(u_j^n - u_{j-1}^n) & a > 0 \\ u_j^n - \frac{a\Delta t}{h}(u_{j+1}^n - u_j^n) & a < 0 \end{cases}$$