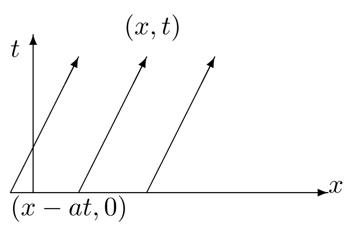
MAT 228B - Feb. 17, 2009

The Lax-Friedrichs method for the linear advection equation, $u_t + au_x = 0$, is stable provided $|a\Delta t/h| \le 1$. This is a common constraint for explicit methods.

CFL condition

In hyperbolic equations, there is a finite speed of propagation. Let's consider this when designing numerical schemes.

The domain of dependence of the point (x,t) is the set of all points that the solution depends on at (x,t).

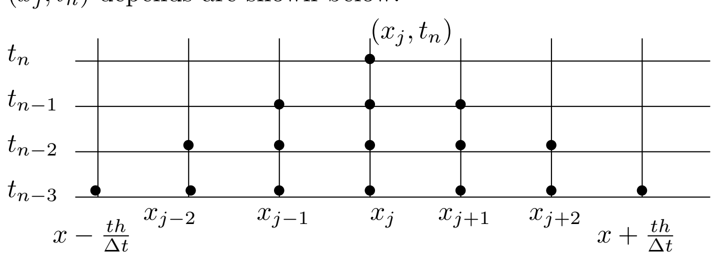


For $u_t + au_x = 0$, the solution is constant along the characteristics $x - at = x_0$.

The domain of dependent(DOD for short) is the single point $(x-at,0)$.

What is DOD of the point (x,t) for $u_t = u_{xx}$? The DOD is $\mathbb{R}$.

For a numerical method, we have the idea of the numerical domain of dependence: the points in discrete space-time on which the discrete solution at (x_j,t_n) depends are shown below.



assume it is a three-point, centered, explicit scheme

The bullets represent the numerical DOD of (x_j,t_n).

What if we refine time and space with $r = \Delta t/h$ constant? The numerical DOD is constrained in the interval $[x - t/r,\ x + t/r]$, as $h, \Delta t \to 0$, with $r = \Delta t/h$fixed

It seems reasonable that for the solution of the numerical scheme to converge to the solution of the PDE, we should have that $x - at \in [x - \frac{t}{r}, x + \frac{t}{r}]$, i.e. the

DOD of the PDE should be contained in the numerical DOD.

$$x - \frac{t}{r} \le x - at \le x + \frac{t}{r}$$
$$-\frac{1}{r} \le a \le \frac{1}{r}$$
$$|ar| \le 1$$
$$\left|\frac{a\Delta t}{h}\right| \le 1$$

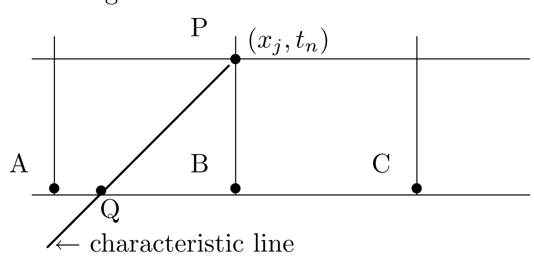
This is a necessary condition for convergence.

CFL condition: If a numerical scheme is convergent, then the domain of dependence of the PDE must be contained in the domain of dependence of the numerical scheme.

Note that the CFL condition is necessary but not sufficient for convergence, e.g. FE centered space is not stable.

Upwind Method

Since the solution to the PDE depends on points on only one side, this motivates us to investigate one-sided schemes. Assume $a > 0$.



Let $P = (x_j, t_n)$. $u_j^n = u(P) = u(Q)$.

How to approximate $u(Q)$?

We can interpolate using $u(A)$ and $u(B)$.

The coordinates of the points A, B, and Q are

$$A = (x_{j-1}, t_{n-1}), \quad B = (x_j, t_{n-1}), \quad Q = (x_j - a\Delta t, t_{n-1}).$$

The length of line segment QA is

$$\begin{aligned}\overline{QA} &= x_j - a\Delta t - x_{j-1} \\ &= h - a\Delta t \\ &= h(1-\nu),\end{aligned}$$

and the length of QB is

$$\overline{QB} = a\Delta t = h\nu.$$

The interpolated value of u at Q is

$$\begin{aligned}u(Q) &= \frac{\overline{QB}}{\overline{AB}}\, u(A) + \frac{\overline{QA}}{\overline{AB}}\, u(B) \\ &= \nu\, u(A) + (1-\nu)\, u(B).\end{aligned}$$

This gives the numerical scheme

$$
\begin{aligned}
u_j^n &= \nu u_{j-1}^{n-1} + (1-\nu) u_j^{n-1} \\
&= u_j^{n-1} - \nu (u_j^{n-1} - u_{j-1}^{n-1}) \\
&= u_j^{n-1} - \frac{a\Delta t}{h} (u_j^{n-1} - u_{j-1}^{n-1}).
\end{aligned}
$$

This can be written as

$$
\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a}{h}, (u_j^n - u_{j-1}^n) = 0
$$

which is forward in time and backward in space. This was derived for $a > 0$. For general a, the Upwind scheme is

$$
u_j^{n+1} = \begin{cases} u_j^n - \frac{a\Delta t}{h} (u_j^n - u_{j-1}^n) & a > 0 \\ u_j^n - \frac{a\Delta t}{h} (u_{j+1}^n - u_j^n) & a < 0 \end{cases}.
$$