MAT 228B - Feb. 19, 2009

Introduction

Continued from last time: upwinding-idea, one-sided spatial differencing for the advection equation

$$u_t + au_x = 0.$$

If $a > 0$ the solution is translating to the right, and if $a < 0$ the solution is translating to the left. In general, the upwind scheme is

$$u_j^{n+1} = \begin{cases} u_j^{n+1} = u_j^n - \dfrac{a\Delta t}{h}(u_j^n - u_{j-1}^n) & \text{if } a > 0 \\ u_j^{n+1} = u_j^n - \dfrac{a\Delta t}{h}(u_{j+1}^n - u_j^n) & \text{if } a < 0 \end{cases}$$

Use the piecewise defined scheme for variable coefficient (a changing sign)

Accuracy of upwind

The LTE=$O(\Delta t) + O(h)$, and so upwind is first-order in space and time. Is it stable? The amplification factor is

$$g(\xi) = 1 - \nu(1 - e^{-i\xi h}) = (1 - \nu) + \nu e^{-i\xi h}.$$

Assume that $a > 0$. Upwinding is stable if $\nu = \frac{a\Delta t}{h} \leq 1$. Thus we can compute up the constraint imposed by the CFL condition.

Getting second order in space and time

Referring to the figure from last time, we chose BCD and interpolate to Q to get Lax-Wendroff method. If we choose ABC and interpolate to Q, we get Beam Warming.

More traditional derivation of these methods is based on the Taylor expansion

$$u(x, t + \Delta t) = u(x, t) + \Delta t u_t(x, t) + \frac{\Delta t^2}{2} u_{tt}(x, t) + O(\Delta t^3).$$

Notice $u_t = -au_x$. Thus

$$u(x, t + \Delta t) = u(x, t) - a\Delta t u_x + O(\Delta t^2).$$

To get higher accuracy, we need to include u_{tt} term. We use the PDE to replace u_{tt} with

$$u_{tt} = \frac{\partial}{\partial t}(-au_x) = -au_{xt} = -a(u_t)_x = -a(-au_x)_x = a^2 u_{xx} \qquad (a = \text{constant}).$$

The Taylor expansion is then

$$u(x, t + \Delta t) = u - a\Delta t u_x + \frac{\Delta t^3 a^2}{2} u_{xx} + O(\Delta t^3).$$

Now discretize the u_x and u_{xx} terms with second-order spatial differences to get a second-order numerical scheme. For Lax-Wendroff use the centered difference for both the first and second spatial difference:

$$u_j^{j+1} = u_j^n - \frac{a\Delta t}{2h}(u_{j+1}^n - u_{j-1}^n) + \frac{a^2\Delta t}{2h^2}(u_{j+1}^n - 2u_j + u_{j-1}^n)$$

The LTE is clearly $O(\Delta t^2) + O(h^2)$

Beam Warming:
We use the one-sided upwind difference for u_x and u_{xx}:

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2h}(3u_j^n - 4u_{j-1}^n + u_{j-2}^n) + \frac{a^2\Delta t^2}{2h^2}(u_j^n - 2u_{j-1}^n + u_{j+2}^n). \quad \text{for } a > 0$$

Note that the one-sided, three-point second difference is only first-order accurate in space. The scheme is still second order accurate because the LTE=$O(\Delta t\, h) + O(h^2)$.

Stability of Lax-Wendroff:
We need $|\nu| \leq 1$ from CFL. The amplification factor is

$$|g(\xi)|^2 = 1 - 4\nu^2(1-\nu^2)\sin^4\left(\frac{\xi h}{2}\right).$$

Need to check the extreme points of sin: $\xi = 0$ and $\xi h = \pi$. Checking the first,

$$g(0)|^2 = 1.$$

Checking the other point

$$g\left(\frac{\pi}{h}\right) = 1 - 4\nu^2(1-\nu^2) = 1 - 4\nu^2 + 4\nu^4 = (1-2\nu^2)^2.$$

The scheme is stable if

$$|1 - 2\nu^2| \leq 1,$$

which simplifies to

$$|\nu| \leq 1.$$

As with the upwind scheme, we may compute up to the limit imposed by the CFL constraint.