MAT 228B - Feb/24/2009

Observations from demo from last class for discontinuous initial data problem

-Upwinding-smoothing of the solution

-LF-less accurate than upwinding, more smearing

-Lax-Wendroff - unphysical oscillations, behind the jump

notable phase shift(phase lag)

-tried smaller time step (h fixed) and the quality of the solution got worse for both methods.

-Δt smaller $\nu = a\Delta t/h$ fixed

LW-still got oscillations, but more localized

Upwinding- less smearing of the solution

-$\nu = 1$ upwinding is an exact solution

Use Modified equaitions to understand this

To solve PDE $\longrightarrow$(discretize) difference equations solution approximates the PDE solution $\longrightarrow$ PDE that approximates the difference equation.

Upwinding for $a > 0$. A schedule for upwinding is:

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{h}(u_j^n - u_{j-1}^n)$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a}{h}(u_j^n - u_{j-1}^n) = 0$$

Let $v(x, t)$ be a (smooth) continuous function that agrees with the true solution of the difference equation.

What equation dose v solve?

$$\frac{v(x, t + \Delta t) - v(x, t)}{\Delta t} + a\left(\frac{v(x, t) - v(x - h, t)}{h}\right) = 0$$

Taylor expansion $\Delta t, h \to 0$ to find a PDE for v.

$$(v_t + \frac{\Delta t}{2} v_{tt} + \frac{\Delta t^2}{6} v_{ttt} + \cdots) + a(v_x - \frac{h}{2} v_{xx} + \frac{h^2}{6} v_{xxx} + \cdots) = 0$$

$$v_t + av_x = (\frac{ah}{2} v_{xx} - \frac{\Delta t}{2} v_{tt}) - (\frac{h^2}{6} v_{xxx} + \frac{\Delta t^2}{6} v_{ttt}) + \cdots$$

$$\frac{ah}{2} v_{xx} - \frac{\Delta t}{2} v_{tt} = O(\Delta t), \frac{h^2}{6} v_{xxx} + \frac{\Delta t^2}{6} v_{ttt} = O(\Delta t^2)$$

assume $\Delta t = O(h)$

Retain terms up to $O(\Delta t)$

$$v_t + av_x = \frac{1}{2}(ahv_{xx} - \Delta t v_{tt}) + O(\Delta t^2)$$

This is what is called the modified equation for upwinding.

Upwinding approximates $u_t + au_x = 0$ to first order $v_t + av_x = 1/2(ahv_{xx} - \Delta t v_{tt})$ to second order.

I will try to eliminate the v_{tt} term:

Take ∂_t of the equation:

$$v_{tt} = -av_{xt} + O(\Delta t)$$

Take ∂_x of the equation and I get:

$$v_{tx} = -av_{xx} + O(\Delta t)$$

$$v_{tt} = a^2 v_{xx} + O(\Delta t)$$

$$v_t + av_x = \frac{1}{2}(ahv_{xx} - \Delta t a^2 v_{xx}) + O(\Delta t^2)$$

$$\begin{aligned} v_t + av_x &= \frac{ah}{2}(1 - \frac{\Delta t a}{h})v_{xx} \\ &= \frac{ah}{2}(1-\nu)v_{xx} \end{aligned}$$

Upwinding solves an advection equation to first order.

Upwinding solves an advection-diffusion equation to second order.

$$v_t + av_x = D_{up} v_{xx}$$

$$D_{up} = \frac{ah}{2}(1-\nu)$$

as $h \to 0$ with ν fixed $D_{up} \to 0$

as $\Delta t \to 0$ with h fixed, increasing the diffusion

The modified equation for Lax-Friedrichs is given by the equation:

$$v_t + av_x = D_{LF} v_{xx}$$

$$\begin{aligned} D_{LF} &= \frac{h^2}{2\Delta t}(1-\nu^2) \\ &= \frac{ha}{2\nu}(1+\nu)(1-\nu) \\ &= \frac{ah}{2}(1-\nu)(1+\nu) \\ &= D_{up}\frac{1+\nu}{\nu} \\ &= D_{up}(1+\frac{1}{\nu}) \geq D_{up} \end{aligned}$$

Note that Lax-Wendroff provides a second order accurate approximation to:

$$u_t + au_x = 0$$

but a third order approximation to:

$$v_t + av_x = \frac{ah^2}{6}(\nu^2 - 1)v_{xxx}$$

$$v_t + av_x = \mu v_{xxx} \tag{1}$$

What does μv_{xxx} do?

This term is dispersive.

We will study equation (1) on the real line using Fourier transform:

$$\hat{v}_t + ai\xi\hat{v} = -i\xi^3\mu\hat{v}$$

$$\hat{v}_t = (-ai\xi - \mu i\xi^3)\hat{v}$$

$$\hat{v}(\xi, t) = \hat{v}(\xi, 0)e^{-(ai\xi + \mu t\xi^3)t}$$

Transform back:

$$\begin{aligned} v(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \hat{v}(\xi, 0)e^{-(ai\xi + \mu t\xi^3)t} e^{i\xi x} d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \hat{v}(\xi, 0)e^{i\xi(x - c(\xi)t)} d\xi \end{aligned}$$

where $c = a + \mu\xi^3$ - phase velocity

if $\mu = 0$ all modes translate at speed a.

if $\mu \neq 0$ the modes translate at different speeds (dispersion).

Assume $a > 0$, then for LW, $\mu < 0$.

-for ξ small, the speed is approximately a

-for ξ large, the modes travel at speed slower than a, $c(\xi) < a$ for $\mu < 0$, $c(\xi)$ is called the speed.

All frequencies have a phase lag.

This explains the oscillations in LW.