MAT 228B - Feb/24/2009

Observations from demo from last class for discontinuous initial data problem

- Upwinding produced smoothing of the solution.
- LF-less accurate than upwinding and more smearing
- Lax-Wendroff produced unphysical oscillations, behind the jump; notable phase shift(phase lag)
- tried smaller time step (h fixed) and the quality of the solution got worse for both methods.
- for Δt smaller $\nu = a\Delta t/h$ fixed
  LW still still produced oscillations, but they were more localized;
  Upwinding gave less smearing of the solution
- for $\nu = 1$ upwinding is an exact solution

We can use modified equations to understand these observations. We introduced the difference equations to approximate the PDE. We now want to understand the solutions of the difference equations. The idea of modified equations is to find a PDE that approximates the difference equations.

Upwinding for $a > 0$:

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{h}(u_j^n - u_{j-1}^n)$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a}{h}(u_j^n - u_{j-1}^n) = 0$$

Let $v(x,t)$ be a (smooth) continuous function that agrees with the true solution of the difference equation. What equation does v solve?

$$\frac{v(x,t+\Delta t) - v(x,t)}{\Delta t} + a\left(\frac{v(x,t) - v(x-h,t)}{h}\right) = 0$$

Taylor expand $\Delta t, h \to 0$ to find a PDE for v.

$$\left(v_t + \frac{\Delta t}{2}v_{tt} + \frac{\Delta t^2}{6}v_{ttt} + \cdots\right) + a\left(v_x - \frac{h}{2}v_{xx} + \frac{h^2}{6}v_{xxx} + \cdots\right) = 0$$

$$v_t + av_x = \left(\frac{ah}{2}v_{xx} - \frac{\Delta t}{2}v_{tt}\right) - \left(\frac{h^2}{6}v_{xxx} + \frac{\Delta t^2}{6}v_{ttt}\right) + \cdots$$

The first term on the right is $= O(\Delta t)$ and the second term is $O(\Delta t^2)$ assuming that $\Delta t = O(h)$. Retaining terms up to $O(\Delta t)$ gives the modified equation for upwinding:

$$v_t + av_x = \frac{1}{2}(ahv_{xx} - \Delta t v_{tt}) + O(\Delta t^2).$$

Upwinding approximates

$$u_t + au_x = 0$$

to first-order, but it approximates

$$v_t + av_x = 1/2(ahv_{xx} - \Delta t v_{tt})$$

to second-order. To understand the behavior of this equation, we eliminate the v_{tt} term as described below. Take ∂_t of the original equation:

$$v_{tt} = -av_{xt} + O(\Delta t)$$

Take ∂_x of the original equation:

$$v_{tx} = -av_{xx} + O(\Delta t).$$

Combining these gives

$$v_{tt} = a^2 v_{xx} + O(\Delta t).$$

Therefore the modified equation for upwinding is

$$v_t + av_x = \frac{1}{2}(ahv_{xx} - \Delta t a^2 v_{xx}) + O(\Delta t^2)$$

To leading order this is

$$\begin{aligned} v_t + av_x &= \frac{ah}{2}(1 - \frac{\Delta t a}{h})v_{xx} \\ &= \frac{ah}{2}(1 - \nu)v_{xx}. \end{aligned}$$

From the above analysis we see that, upwinding solves an advection equation to first order and it solves an advection-diffusion equation to second order.

$$v_t + av_x = D_{up} v_{xx},$$

where

$$D_{up} = \frac{ah}{2}(1 - \nu).$$

Note that as $h \to 0$ with ν fixed $D_{up} \to 0$, and as $\Delta t \to 0$ with h fixed, the diffusion in upwinding increases. This is consistent with our observations.

The modified equation for Lax-Friedrichs is to leading order

$$v_t + av_x = D_{LF} v_{xx},$$

where the diffusion coefficient is

$$\begin{aligned}
D_{LF} &= \frac{h^2}{2\Delta t}(1-\nu^2) \\
&= \frac{ha}{2\nu}(1+\nu)(1-\nu) \\
&= \frac{ah}{2}(1-\nu)(1+\nu) \\
&= D_{up}\frac{1+\nu}{\nu} \\
&= D_{up}(1+\frac{1}{\nu}) \geq D_{up}
\end{aligned}$$

Thus Lax-Fridrichs is more diffusive that upwinding.

Lax-Wendroff provides a second-order accurate approximation to the advection equation,

$$u_t + au_x = 0,$$

but a third-order approximation to

$$\begin{aligned}
v_t + av_x &= \frac{ah^2}{6}(\nu^2-1)v_{xxx} \\
v_t + av_x &= \mu v_{xxx}. \qquad (1)
\end{aligned}$$

What does μv_{xxx} do? This term is dispersive. We will study equation (1) on the real line using Fourier transform:

$$\hat{v_t} + ai\xi\hat{v} = -i\xi^3\mu\hat{v}$$

$$\hat{v_t} = (-ai\xi - \mu i\xi^3)\hat{v}$$

$$\hat{v}(\xi,t) = \hat{v}(\xi,0)e^{-(ai\xi+\mu t\xi^3)t}$$

Transform back:

$$\begin{aligned}
v(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} \hat{v}(\xi,0)e^{-(ai\xi+\mu t\xi^3)t}e^{i\xi x}d\xi \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} \hat{v}(\xi,0)e^{i\xi(x-c(\xi)t)}d\xi
\end{aligned}$$

where $c = a + \mu\xi^3$ is called the phase velocity. If $\mu = 0$ all modes translate at speed a. If $\mu \neq 0$ the modes translate at different speeds (dispersion).

Assume $a > 0$, then for LW, $\mu < 0$. For ξ small (low frequencies), the phase velocity is approximately a. For ξ large (high frequencies), the modes travel at a speed slower than a All frequencies have a phase lag. This explains the oscillations in LW.