MAT 228B - February 26, 2009

Analysis of Lax-Wendroff and upwinding using Modified Equations

Modified equation for Lax-Wendroff

$$v_t + av_x = \mu v_{xxx}$$
$$\mu = \frac{ah^2}{6}(\nu^2 - 1) < 0 \longrightarrow \text{ phase lag}$$

LW works well for smooth problems
- dispersive oscillations are only a problem when the solution has sharp gradients.

• For $u(x)$ in C^∞, $\hat{u}(\xi)$ decays exponentially as $|\xi| \to \infty$
• If $u(x)$ has a jump discontinuity, $\hat{u}(\xi)$ decays like $1/|\xi|$ as $|\xi| \to 0$

In nonlinear hyperbolic equations, the solution can form discontinuities even for C^∞ initial data. (e.g. shock)

If we include the next order term

$$v_t + av_x = \mu v_{xxx} - \epsilon \underbrace{v_{xxxx}}\to \text{ damping high frequencies}$$
$$\epsilon = O(h^3)$$

- LW is dissapative order 4
- Upwinding is dissapative order 2

Accuracy with a discontinuous solution
- use modified equations

Modified equation for upwinding

$$v_t + av_x = Dv_{xx}$$

Solve $v_t + av_x = 0$ and $v_t + av_x = Dv_{xx}$ on the whole real line with initial data:

$$u(x,0) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x \geq 0. \end{cases}$$

Solutions are:

$$u(x,t) = u(x - at, 0)$$
$$v(x,t) = 1 - erf\left(\frac{(x-at)}{\sqrt{4Dt}}\right) \text{ , where } erf(z) = \frac{2}{\sqrt{\pi}}\int_{-\infty}^{z} \exp{(-s^2)}ds$$

Apply the 1-norm:

$$\begin{aligned}
\|u(x,t) - v(x,t)\|_1 &= \int_{-\infty}^{\infty} |u(x,t) - v(x,t)| dx \\
&= \int_{-\infty}^{\infty} \left| u(x-at,0) - \left(1 - erf\left(\frac{(x-at)}{\sqrt{4Dt}}\right)\right)\right| dx \\
\text{let } z = x - at, & \\
&= \int_{-\infty}^{0} \left| erf\left(\frac{z}{\sqrt{4Dt}}\right)\right| dz + \int_{0}^{\infty} \left| 1 - erf\left(\frac{z}{\sqrt{4Dt}}\right)\right| dz \\
&= 2\int_{-\infty}^{0} erf\left(\frac{z}{\sqrt{4Dt}}\right) dz
\end{aligned}$$

Let $s = \frac{z}{\sqrt{4Dt}}$, then

$$\begin{aligned}
\|u - v\|_1 &= 2\sqrt{4Dt}\int_{-\infty}^{0} erf(s) ds \\
&= C\sqrt{Dt}
\end{aligned}$$

where C is independent of t and D. Plugging in $D = \frac{ah}{2}(1-\nu)$, we find that:

$$\|u - v\|_1 = O(\sqrt{h}).$$

- good news : this converges as $h \to 0$, ν fixed
- bad news : less than first order accuracy

Analysis of LW and upwinding using Fourier series

In addition to modified equations, we can analyze using Fourier series
Von Neumann Analysis:

$$\begin{aligned}
g_{up}(\xi) &= 1 - \nu(1 - e^{-i\xi h}) \\
g_{LW}(\xi) &= 1 - i\nu\sin(\xi h) - 2\nu^2\sin^2\left(\frac{\xi h}{2}\right)
\end{aligned}$$

[Note: amplitudes in Fourier space are preserved in the analytic solution.]

$$|g_{up}(\xi)| = 1 - \frac{1}{2}(\nu - \nu^2)(\xi h)^2 + O((\xi h)^4)$$

this has an amplitude error of $O((\xi h)^2)$

$$|g_{LW}(\xi)| = 1 - \frac{1}{8}(\nu^2 - \nu^4)(\xi h)^4 + O((\xi h)^6)$$

this is fourth order in the amplitude

The Fourier mode

$$u = e^{i(\xi x + \omega t)}$$

is a solution to $u_t + au_x = 0$ if $\omega = -a\xi$ (dispersion relation).

In one time step, the phase changes by $\omega \Delta t = -a\xi \Delta t$.

The $\arg(g(\xi))$ = the phase change per time step in the numerical scheme.

Define the relative phase error as: $\dfrac{\arg(g(\xi))}{\omega \Delta t}$

$$\text{upwind: } 1 - \frac{1}{6}(1-\nu)(1-2\nu)(\xi h)^2 + O(h^2)$$
$$\text{Lax-Wendroff: } 1 - \frac{1}{6}(1-\nu^2)(\xi h)^2 + O(h^2)$$

- both have $O(h^2)$ phase error and upwinding has smaller error

[See handout for comparison of the two schemes]