# MAT 228B - February 26, 2009

## Analysis of Lax-Wendroff and upwinding using Modified Equations

The modified equation for Lax-Wendroff is

$$\begin{aligned} v_t + av_x &= \mu v_{xxx} \\ \mu &= \frac{ah^2}{6}(\nu^2 - 1) < 0 \longrightarrow \text{ phase lag} \end{aligned}$$

LW works well for smooth problems, but it shows dispersive oscillations when the solution has sharp gradients.

- For $u(x)$ in $C^\infty$, $\hat{u}(\xi)$ decays exponentially as $|\xi| \to \infty$
- If $u(x)$ has a jump discontinuity, $\hat{u}(\xi)$ decays like $1/|\xi|$ as $|\xi| \to 0$

In nonlinear hyperbolic equations, the solution can form discontinuities even for $C^\infty$ initial data. (e.g. shock).

Lax-Wendroff does have some dissipation. If we include the next order term in the modified equation we get

$$\begin{aligned} v_t + av_x &= \mu v_{xxx} - \epsilon \underbrace{v_{xxxx}} \to \text{ damping high frequencies} \\ \epsilon &= O(h^4) \end{aligned}$$

- LW is dissipative order 4
- Upwinding is dissipative order 2

**Accuracy with a discontinuous solution**

The modified equation for upwinding is

$$v_t + av_x = Dv_{xx}$$

Solve $u_t + au_x = 0$ and $v_t + av_x = Dv_{xx}$ on the whole real line with initial data:

$$u(x,0) = v(x,0) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x \geq 0. \end{cases}$$

The solutions are

$$\begin{aligned} u(x,t) &= u(x - at, 0) \\ v(x,t) &= 1 - erf\left(\frac{(x-at)}{\sqrt{4Dt}}\right), \end{aligned}$$

where

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{z} \exp(-s^2) ds.$$

The 1-norm of the difference in these two functions is

$$\begin{aligned}\|u(x,t)-v(x,t)\|_1 &= \int_{-\infty}^{\infty} |u(x,t)-v(x,t)|dx \\ &= \int_{-\infty}^{\infty} \left|u(x-at,0)-\left(1-erf\left(\frac{(x-at)}{\sqrt{4Dt}}\right)\right)\right| dx\end{aligned}$$

Let $z = x - at$, so that

$$\begin{aligned}\|u(x,t)-v(x,t)\|_1 &= \int_{-\infty}^{0} \left|erf\left(\frac{z}{\sqrt{4Dt}}\right)\right| dz + \int_{0}^{\infty} \left|1-erf\left(\frac{z}{\sqrt{4Dt}}\right)\right| dz \\ &= 2\int_{-\infty}^{0} erf\left(\frac{z}{\sqrt{4Dt}}\right) dz\end{aligned}$$

Let $s = \frac{z}{\sqrt{4Dt}}$, then

$$\begin{aligned}\|u-v\|_1 &= 2\sqrt{4Dt}\int_{-\infty}^{0} erf(s)ds \\ &= C\sqrt{Dt}\end{aligned}$$

where $C$ is independent of $t$ and $D$. Plugging in $D = \frac{ah}{2}(1-\nu)$, we find that:

$$\|u-v\|_1 = O(\sqrt{h}).$$

- good news : this converges as $h \to 0$, $\nu$ fixed
- bad news : less than first order accuracy

## Analysis of LW and upwinding using Fourier series

In addition to modified equations, we can analyze using Fourier series
**Von Neumann Analysis:**

$$\begin{aligned}g_{up}(\xi) &= 1-\nu(1-e^{-i\xi h}) \\ g_{LW}(\xi) &= 1-i\nu\sin(\xi h)-2\nu^2\sin^2\left(\frac{\xi h}{2}\right)\end{aligned}$$

[Note: amplitudes in Fourier space are preserved in the analytic solution.]

$$|g_{up}(\xi)| = 1-\frac{1}{2}(\nu-\nu^2)(\xi h)^2+O((\xi h)^4)$$

Upwinding has an amplitude error of $O((\xi h)^2)$.

$$|g_{LW}(\xi)| = 1-\frac{1}{8}(\nu^2-\nu^4)(\xi h)^4+O((\xi h)^6)$$

Lax-Wendroff is fourth order in the amplitude.

The Fourier mode

$$u = e^{i(\xi x + \omega t)}$$

is a solution to $u_t + au_x = 0$ if $\omega = -a\xi$ (dispersion relation). In one time step, the phase changes by $\omega \Delta t = -a\xi\Delta t$. $\arg(g(\xi))$ = the phase change per time step in the numerical scheme. Define the relative phase error as

$$\frac{\arg(g(\xi))}{\omega \Delta t}.$$

For low frequencies, the relative phase error is

$$\begin{aligned} \text{upwind: } & 1 - \frac{1}{6}(1-\nu)(1-2\nu)(\xi h)^2 + O(h^2) \\ \text{Lax-Wendroff: } & 1 - \frac{1}{6}(1-\nu^2)(\xi h)^2 + O(h^2) \end{aligned}$$

Both methods have an $O(h^2)$ phase error and upwinding has smaller error.
[See handout for comparison of the two schemes]