MAT 228B - March 3, 2009

Boundary Condition

Solve

$$u_t + au_x = 0 \; on[0,1]$$

What boundary condition do I need? Assume $a > 0$. Solution is constant along the characteristic curves

$$x - at = C$$

For $a > 0$, the solution is determined at $x = 1$ (outflow boundary) by initial data or data at $x = 0$
Need a boundary condition at $x = 0$ (inflow boundary).

Try Upwinding,

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{h}(u_j^n - u_{j-1}^n)$$

No problem with boundary condition beacuse the update at $j = N+1$ involves data at $j = N+1$ and $j = N$

Try Lax-Wendroff,

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2h}(u_(j+1)^n - u_{j-1}^n) + \frac{a^2\Delta t^2}{2h^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

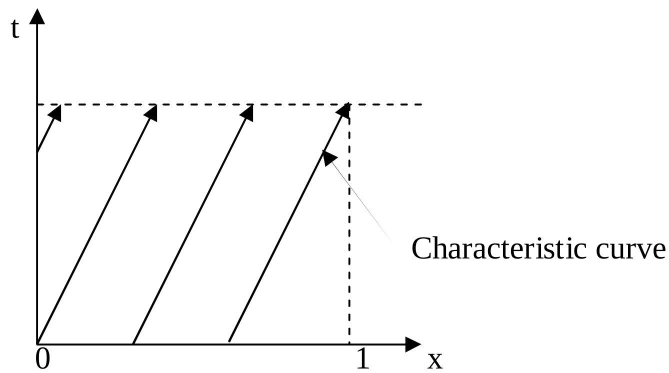
We cannot use this equation at $j = N + 1$ (right boundary). There is no BC on the right for the PDE, but L-W needs a BC at the right.

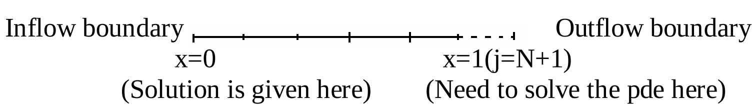
What to do?
1. Use a one sided scheme at the boundary (Upwinding or Beam-Warming)
2. Use a numerical BC, e.g., extrapolation.
   - -Constant extrapolation
     ($u_{N+2}^n = u_{N+1}^n$)
   - -Linear extrapolation
     $u_{N+2}^n = 2u_{N+1}^n - u_N^n$
   - -Use the PDE to extrapolate.

We want to avoid degrading the accuracy or quality of the solution upstream so that no reflection occurs back into the domain.

System of PDE's

$$\underline{u}_t + A\underline{u}_x = 0$$
$$\underline{u}_{tt} = c^2\underline{u}_{xx}$$


t
Characteristic curve
0
1
x


Inflow boundary
Outflow boundary
x=0
x=1(j=N+1)
(Solution is given here)
(Need to solve the pde here)

$$\underline{q} = \begin{pmatrix} u_t \\ u_x \end{pmatrix}$$

$$\underline{q}_t + \begin{pmatrix} 0 & c^2 \\ 1 & 0 \end{pmatrix} \underline{q}_x = 0$$

Lax-Wendroff for a system,

$$\underline{u}_j^{n+1} = \underline{u}_j^n - \frac{A\Delta t}{2h}(\underline{u}_{j+1}^n - \underline{u}_{j-1}^n) + \frac{A^2\Delta t^2}{2h^2}(\underline{u}_{j+1}^n - 2\underline{u}_j^n + \underline{u}_{j-1}^n)$$

How to upwind if the waves are propagating in different direction?
Assume A is constant which is diagonalizable and has full set of real eigenvalues.

$$A = R\Omega R^{-1}$$

where Ω is diagonal matrix which diagonals are the wave speeds (eigen values of A).

$$\underline{u}_t + R\Omega R^{-1}\underline{u}_x = 0$$
$$R^{-1}\underline{u}_t + \Omega R^{-1}\underline{u}_x = 0$$

Define $w = R^{-1}\underline{u}$
$\underline{w}_t + \Omega \underline{w}_x = 0$ - so we get N decoupled scalar equations.

w is called the characteristic co-ordinate.

This system is stable if for all eigen values of A,

$$\left|\frac{\lambda \Delta t}{h}\right| < 1$$

λ are the eigen values of A.

$$\Omega = \underbrace{\Omega^+}_{\text{Positive eigen values}} + \underbrace{\Omega^-}_{\text{Negative eigen values}}$$

$$|\Omega| = \begin{pmatrix} |\lambda_1| & 0 & 0 & 0 & 0 \\ 0 & |\lambda_2| & 0 & 0 & 0 \\ 0 & 0 & |\lambda_3| & 0 & 0 \\ 0 & 0 & 0 & |\lambda_4| & 0 \\ 0 & 0 & 0 & 0 & |\lambda_n| \end{pmatrix}$$

$$\Omega^+ = \frac{(\Omega + |\Omega|)}{2}$$
$$\Omega^- = \frac{(\Omega - |\Omega|)}{2}$$

$$\underline{w}_j^{n+1} = \underline{w}_j^n - \underbrace{\frac{\Delta t}{h}\Omega^+(\underline{w}_j^n - \underline{w}_{j-1}^n)}_{\text{Right moving wave}} - \underbrace{\frac{\Delta t}{h}\Omega^-(\underline{w}_{j+1}^n - \underline{w}_j^n)}_{\text{Left moving wave}}$$

Transform back to $\underline{u}$

$$\underline{u}_j^{n+1} = \underline{u}_j^n - \frac{\Delta t}{h} A^+ (\underline{u}_j^n - \underline{u}_{j-1}^n) - \frac{\Delta t}{h} A^- (\underline{u}_{j+1}^n - \underline{u}_j^n)$$
$$A^+ = R\Omega^+ R^{-1}, A^- = R\Omega^- R^{-1}$$

Fractional stepping method is applicable for this type of system of PDEs in multidimension.

Conservation Law

$$u_t + (f(u))_x = 0$$

Here, f is the flux function. In advection equation $f(u) = au$
This PDE comes from the integral form of the conservative law;

$$\frac{d}{dt} \int_{x_1}^{x_2} u(x,t)dx = f(u(x_1,t)) - f(u(x_2 t))$$

The amount of u in $[x_1, x_2]$ only changes from flux of u across the boundary.

In finite difference methods,

$$u_j = u(x_j)$$

$\leftarrow$ Sampling or approximating a continuos function.

Finite Volume Method

- Discretize space into a set of volumes rather than a set of points and approximate the function by their averages over those volumes.

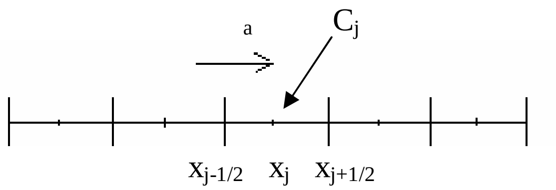
In $1 - D$, divide the domain into cells (intervals).

$$C_j = [x_{j-1/2}, x_{j+1/2}]$$
$$x_j = jh$$

u_j= Average value of $u(x)$ over cell j.

$$u_j = \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x)dx = u(x_j) + O(h^2)$$

$x_j = jh$ is the center of the cell. This leads to cell centered mesh.


a
C_j
x_{j-1/2}
x_j
x_{j+1/2}