Math 228B, Lecture 6
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Homework has been posted on the web and is due two weeks from last Tuesday.

**Von Neumann Analysis — Stability Analysis in Fourier Spaces**

Remember what is stability? A linear method of the form

$$u^{n+1} = Bu^n + b^n$$

is Lax-Richtmyer stable if for each time $T$, there is a constant $C_T > 0$, independent of $\Delta t$, such that $||B^n|| \leq C_T$ for all $n\Delta t \leq T$.

Last time we did two examples where we showed $||B|| \leq 1$ and hence $||B^n|| \leq 1$. But this is too strong a condition for all problems, because what if the solution is supposed to be growing in time rather than decaying?

In general we could show: If there is a constant $\alpha > 0$ independent of $\Delta t$, such that $||B|| \leq 1 + \alpha\Delta t$, then the scheme is stable. Proof:

$$||B^n|| \leq ||B||^n \leq (1 + \alpha\Delta t)^n \leq e^{\alpha\Delta t n} \leq e^{\alpha T}$$

Second to last step: Think as the first two terms in a Taylor expansion about $\Delta t = 0$.
We actually allow growth, and we even allow exponential growth with growth rate $\alpha$.
Consider

$$u_t = bu_{xx} + cu, \qquad b \geq 0$$

Could have a problem where we are growing away from the unstable steady state.

- If $c > 0$, solution may grow.
- If $c < 0$, solution decays.

**Stability Analysis Using Forward Euler**

$$u^{n+1} = (I + \Delta t b L + c\Delta t I)u^n$$

Show this satisfies stability under infinity norm regardless of choice of $c$.

$$||B||_\infty = \frac{b\Delta t}{h^2} + \left|1 + \frac{b\Delta t}{h^2}(-2) + c\Delta t\right| + \frac{b\Delta t}{h^2}$$

Assume (as we did last time) that

$$1 - 2\frac{b\Delta t}{h^2} \geq 0$$

Using the triangle inequality we have

$$||B||_\infty \leq 2\frac{b\Delta t}{h^2} + \left|1 - 2\frac{b\Delta t}{h^2}\right| + |c|\,\Delta t = 1 + |c|\Delta t$$

Therefore this is stable for all $c$ provided that $1 - 2\frac{b\Delta t}{h^2} \geq 0$.

If there was no diffusion term, the PDE would just grow exponentially. Our analysis has allowed growth. So this is fine for $c > 0$, but what if $c < 0$? The solution should not be growing.

Assume the domain is $[0, 1]$ with Dirichlet Boundary Conditions.

$$B = (1 + \Delta t c)I + \Delta t b L$$

$\lambda_k = k^{th}$ eigenvalue of L, $\mu_k = k^{th}$ eigenvalue of B, then $\mu_k = (1 + \Delta t c) + \Delta t b \lambda_k$.

Assume $c < 0$, then both terms are negative, we have

$$\mu_k = 1 + \Delta t(c + b\lambda_k).$$

To prevent growth, we need that $|\mu_k| \leq 1$ for all $k$, i.e.

$$
\begin{aligned}
-1 &\leq 1 + \Delta t(c + b\lambda_k) \leq 1, \\
-2 &\leq \Delta t(c + b\lambda_k) \leq 0, \\
\Delta t &\leq \frac{-2}{c + b\lambda_k}.
\end{aligned}
$$

If $c = 0$, we enforce the same constraint as we previously discussed. But otherwise, this constraint is tighter than the previous derived bound on the time step. This version is called *Strong Stability*.

Note that we still call the previous notion stable because the solution will not grow as we refine the number of time steps (i.e. as $\Delta t \to 0$). But this does not tell us about convergence. If you are actually writing code, you should check your time step satisfies this second (Strong) stability constraint. So this version is also called *Practical Stability*.
Moral: Need to allow growth because problem might be growing, but we don't want the numerical solution to grow if the solution does not actually grow.

**Von Neumann Analysis**

Motivation: In general it can be difficult to estimate the norms of a matrix. So we will analyze Fourier series instead.
On day 2 we used Fourier transforms to solve PDE's on the whole real line. We can do this with any constant coefficient, linear PDE on the whole real line.
Can use the same ideas to analyze linear, constant coefficient difference equations on the infinite lattice $X_j = jh, j \in \mathbb{Z}$. Or can also do this on periodic domains. $e^{i\xi x_j}$ are eigenfunctions of linear, constant, coefficients of difference operators.

For example, for the centered difference operator:

$$
\frac{e^{i\xi x_{j+1} - e^{i\xi x_{j-1}}}}{2h} = e^{i\xi x_j} \frac{e^{i\xi h} - e^{-i\xi h}}{2h} = e^{i\xi x_j} \frac{2i \sin(\xi h)}{2h},
$$

the eigenvalue is $\frac{i}{h}\sin(\xi h)$. And the Second Derivative operator:

$$
\frac{e^{i\xi x_{j-1}} - 2e^{i\xi x_j} + e^{i\xi x_{j+1}}}{h^2} = e^{i\xi x_j} \left( \frac{2}{h^2}(\cos(\xi h) - 1) \right) = \frac{-4}{h^2} \sin^2(\frac{\xi h}{2}) e^{i\xi x_j}
$$

Let $v_j$ be a grid function. The Fourier Transform of $v_j$ is

$$
\hat{v}(\xi) = \frac{h}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} v_j e^{-i\xi x_j}, \qquad \text{for} - \pi \leq \xi h \leq \pi
$$

This doesn't make sense for $\xi \in (-\infty, \infty)$ since we are on the finite lattice — high frequencies (with large wave number $\xi$) cannot be represented on a finite grid.

The Inverse Transform is

$$
v_j = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} \hat{v}(\xi) e^{i\xi x_j} \, d\xi.
$$

And we have Parseval's Relation (Discrete Version)

$$
||v||_2 = ||\hat{v}||_2,
$$

but note

$$
||v||_2 = \left( h \sum_{j=-\infty}^{\infty} |v_j|^2 \right)^{1/2}
$$

We have an analogy here:

| | | |
|---|---|---|
| x space | $\leftrightarrow$ | $\xi$ space |
| finite | $\leftrightarrow$ | discrete (Fourier Series) |
| discrete | $\leftrightarrow$ | finite |

Consider homogeneous equation

$$
u^{n+1} = Bu^n
$$

This is stable if $||B||_2 \leq 1 + \alpha\Delta t$

$$
||u^{n+1}||_2 \leq (1 + \alpha\Delta t)||u^n||_2
$$

Use Parseval's Relation, and instead look at the Fourier transform of the solution in space to show

$$||\hat{u}^{n+1}||_2 \leq (1+\alpha\Delta t)||\hat{u}^n||_2$$

**Example: Forward Euler for Diffusion Equation on Real Line**
The discretization scheme is

$$\begin{aligned}\frac{u_j^{n+1}-u_j^n}{\Delta t} &= b\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2} \\ \Rightarrow u_j^{n+1} &= u_j^n + \frac{b\Delta t}{h^2}(u_{j-1}^n - 2u_j^n + u_{j+1}^n)\end{aligned}$$

where

$$u_j^n = \frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h} e^{ix_j\xi}\hat{u}^n(\xi)d\xi.$$

Hence, we have

$$\begin{aligned}u_j^{n+1} &= \frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h}\hat{u}^n(\xi)\Big[e^{ix_j\xi} + \frac{b\Delta t}{h^2}(e^{ix_{j-1}\xi} - 2e^{ix_j\xi} + e^{ix_{j+1}\xi})\Big]d\xi \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h}(\hat{u}^n(\xi)(1 - \frac{4b\Delta t}{h^2}\sin^2(\frac{\xi h}{2}))e^{ix_j\xi}d\xi\end{aligned}$$

Note that

$$\begin{aligned}u_j^{n+1} &= \frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h}\hat{u}^n(\xi)e^{ix_j\xi}d\xi \\ u_j^{n+1}(\xi) &= 1 - \frac{4b\Delta t}{h^2}\sin^2(\frac{\xi h}{2}))\hat{u}^n(\xi)\end{aligned}$$

The above equation has the following form

$$u_j^{n+1}(\xi) = g(\xi)\hat{u}^n(\xi)$$

where $g(\xi)$ is called the amplification factor. For Stability, we need $|g(\xi)| = 1+\alpha\Delta t$ for all $\xi$. Then we have

$$||\hat{u}^{n+1}||_2 \leq (1+\alpha\Delta t)||\hat{u}^n||_2$$

and then

$$||u^{n+1}||_2 = (1+\alpha\Delta t)||u^n||_2$$

$$\begin{aligned}|g(\xi)| = \left|1 - \frac{4b\Delta t}{h^2}\sin^2(\xi h)\right| &\leq 1 \\ -1 \leq 1 - \frac{4b\Delta t}{h^2}\sin^2(\frac{\xi h}{2}) &\leq 1 \\ -2 \leq -\frac{4b\Delta t}{h^2}\sin^2(\frac{\xi h}{2}) &\leq 0 \\ \Delta t \leq \frac{2}{\frac{4b}{h^2}\sin^2(\frac{\xi h}{2})} &= \frac{h^2}{2b\sin^2(\frac{\xi h}{2})}\end{aligned}$$

What is the most severe restriction that we must have? Need $\Delta t \leq \frac{h^2}{2b}$.
To perform Von Neumann Analysis, you assume solution of the form

$$u_j^n = e^{i\xi x_j}, u_j^{n+1} = g(\xi)e^{i\xi x_j}$$

Plug in equation and calculate $g(\xi)$. Looking at Fourier modes is enough to tell you about the stability of the solution.

More generally, if you have a multilevel scheme

$$u_j^n = g(\xi)^n e^{ix_j\xi}$$

**Example:** Leap-Frog (3-level scheme where you do centered difference in time) is unstable for diffusion equation by Von Neumann Analysis. The discretization gives

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = b\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2}$$
$$\frac{g^{n+1}(\xi) - g^{n-1}(\xi)}{2\Delta t} = \frac{-4b}{h^2} g^n(\xi) \sin^2(\frac{\xi h}{2})$$

Which we can then simplify to obtain a quadratic equation for g.

$$g^2 + \frac{8\Delta tb}{h^2} g - 1 = 0.$$

$$g_{+/-} = \frac{-4\Delta tb}{h^2} \sin^2(\frac{\xi h}{2}) \pm \left( \left( \frac{4\Delta tb}{h^2} \sin^2(\frac{\xi h}{2}) \right)^2 + 1 \right)^{1/2}$$

Note that $|g_-| > 1$ for some $\xi$ and thus the scheme is unstable.