Math 228B, Lecture 6
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Von Neumann Analysis — Stability Analysis in Fourier Spaces

Recall the definition of stability. A linear method of the form

$$u^{n+1} = Bu^n + b^n$$

is Lax-Richtmyer stable if for each time T, there is a constant $C_T > 0$, independent of Δt, such that $||B^n|| \leq C_T$ for all $n\Delta t \leq T$.

Last time we did two examples where we showed $||B|| \leq 1$ and hence $||B^n|| \leq 1$. But this is too strong a condition for all problems. What if the solution is supposed to be growing in time rather than decaying?

If there is a constant $\alpha > 0$ independent of Δt, such that $||B|| \leq 1 + \alpha\Delta t$, then the scheme is stable. Proof:

$$||B^n|| \leq ||B||^n \leq (1+\alpha\Delta t)^n \leq e^{\alpha\Delta t n} \leq e^{\alpha T}$$

Second to last step: Think as the first two terms in a Taylor expansion of $\exp(\alpha\Delta t)$ about $\Delta t = 0$. We actually allow growth, and we even allow exponential growth with growth rate α. Consider

$$u_t = bu_{xx} + cu, \qquad b \geq 0$$

- If $c > 0$, solution may grow.
- If $c < 0$, solution decays.

Stability Analysis Using Forward Euler

$$u^{n+1} = (I + \Delta t bL + c\Delta t I)u^n$$

We will show that the stability restriction is the same for this equation as it is for the diffusion equation, regardless of choice of c.

$$||B||_\infty = \frac{b\Delta t}{h^2} + \left|1 + \frac{b\Delta t}{h^2}(-2) + c\Delta t\right| + \frac{b\Delta t}{h^2}.$$

Assume (as we did last time) that

$$1 - 2\frac{b\Delta t}{h^2} \geq 0.$$

Using the triangle inequality we have that

$$||B||_\infty \leq 2\frac{b\Delta t}{h^2} + \left|1 - 2\frac{b\Delta t}{h^2}\right| + |c|\,\Delta t = 1 + |c|\Delta t$$

Therefore this is stable for all c provided that $1 - 2\frac{b\Delta t}{h^2} \geq 0$.

If there were no diffusion ($b = 0$), the solution to the PDE would just grow (or decay) exponentially. Our analysis has allowed growth, and so this is fine for $c > 0$, but what if $c < 0$? The solution should not be growing.

Assume the domain is $[0,1]$ with Dirichlet Boundary Conditions.

$$B = (1 + \Delta tc)I + \Delta t bL$$

Let $\lambda_k = k^{th}$ eigenvalue of L, and $\mu_k = k^{th}$ eigenvalue of B. Then $\mu_k = (1 + \Delta tc) + \Delta t b\lambda_k$.

Assume $c < 0$. Then both terms are negative, and we have

$$\mu_k = 1 + \Delta t(c + b\lambda_k).$$

To prevent growth, we need that $|\mu_k| \leq 1$ for all k, i.e.

$$\begin{aligned} -1 &\leq 1 + \Delta t(c + b\lambda_k) \leq 1, \\ -2 &\leq \Delta t(c + b\lambda_k) \leq 0, \\ \Delta t &\leq \frac{-2}{c + b\lambda_k}. \end{aligned}$$

If $c = 0$, we enforce the same constraint as we previously discussed. But otherwise, this constraint is tighter than the previous derived bound on the time step. This version is called *Strong Stability*.

Note that we still call the previous notion stable because the growth of the solution is bounded as time is refined (i.e. as $\Delta t \to 0$ with $T = n\Delta t$ fixed). This gives information about convergence, but does not address whether the solution is growing for a fixed value of Δt. In practice, you should check your time step satisfies this second (Strong) stability constraint. So this version is also called *Practical Stability*.

Moral: Need to allow growth because problem might be growing, but we don't want the numerical solution to grow if the solution does not actually grow.

Von Neumann Analysis

Motivation: In general it can be difficult to estimate the norms of a matrix. We will analyze the scheme in Fourier space instead.

On day 2 we used Fourier transforms to solve PDE's on the whole real line. We can do this with any constant coefficient, linear PDE on the whole real line.

Can use the same ideas to analyze linear, constant coefficient difference equations on the infinite lattice $X_j = jh, j \in \mathbb{Z}$. Or can also do this on periodic domains. $e^{i\xi x_j}$ are eigenfunctions of linear, constant, coefficients of difference operators.

For example, for the centered difference operator:

$$\frac{e^{i\xi x_{j+1} - e^{i\xi x_{j-1}}}}{2h} = e^{i\xi x_j}\frac{e^{i\xi h} - e^{-i\xi h}}{2h} = e^{i\xi x_j}\frac{2i\sin(\xi h)}{2h},$$

the eigenvalue is $\frac{i}{h}\sin(\xi h)$. And the Second Derivative operator:

$$\frac{e^{i\xi x_{j-1}} - 2e^{i\xi x_j} + e^{i\xi x_{j+1}}}{h^2} = e^{i\xi x_j}\left(\frac{2}{h^2}(\cos(\xi h) - 1)\right) = \frac{-4}{h^2}\sin^2\left(\frac{\xi h}{2}\right)e^{i\xi x_j}.$$

Let v_j be a grid function. The Fourier Transform of v_j is

$$\hat{v}(\xi) = \frac{h}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty} v_j e^{-i\xi x_j}, \qquad \text{for } -\pi \le \xi h \le \pi$$

Note the restriction on ξ — high frequencies (with large wave number ξ) cannot be represented on a finite grid.

The Inverse Transform is

$$v_j = \frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h} \hat{v}(\xi)e^{i\xi x_j}\,d\xi.$$

And we have Parseval's Relation (Discrete Version)

$$||v||_2 = ||\hat{v}||_2,$$

but note that the 2-norm on v is the grid function norm

$$||v||_2 = \left(h\sum_{j=-\infty}^{\infty}|v_j|^2\right)^{1/2}.$$

We have an analogy here:

x space	ξ space
finite	discrete (Fourier Series)
discrete	finite

Consider homogeneous equation

$$u^{n+1} = Bu^n$$

This is stable if $||B||_2 \le 1 + \alpha\Delta t$, so that

$$||u^{n+1}||_2 \le (1 + \alpha\Delta t)||u^n||_2.$$

Use Parseval's Relation, and instead look at the Fourier transform of the solution to show that

$$||\hat{u}^{n+1}||_2 \le (1 + \alpha\Delta t)||\hat{u}^n||_2.$$

Example: Forward Euler for Diffusion Equation on Real Line

The discretization scheme is

$$\begin{aligned}\frac{u_j^{n+1}-u_j^n}{\Delta t} &= b\frac{u_{j-1}^n-2u_j^n+u_{j+1}^n}{h^2}\\ \Rightarrow u_j^{n+1} &= u_j^n+\frac{b\Delta t}{h^2}(u_{j-1}^n-2u_j^n+u_{j+1}^n)\end{aligned}$$

where

$$u_j^n=\frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h}e^{ix_j\xi}\hat{u}^n(\xi)d\xi.$$

Hence, we have

$$\begin{aligned}u_j^{n+1} &= \frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h}\hat{u}^n(\xi)\Big[e^{ix_j\xi}+\frac{b\Delta t}{h^2}(e^{ix_{j-1}\xi}-2e^{ix_j\xi}+e^{ix_{j+1}\xi})\Big]d\xi\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h}\hat{u}^n(\xi)\left(1-\frac{4b\Delta t}{h^2}\sin^2\left(\frac{\xi h}{2}\right)\right)e^{ix_j\xi}d\xi\end{aligned}$$

Note that

$$u_j^{n+1}=\frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h}\hat{u}^{n+1}(\xi)e^{ix_j\xi}d\xi,$$

and so

$$\hat{u}^{n+1}(\xi)=\left(1-\frac{4b\Delta t}{h^2}\sin^2\left(\frac{\xi h}{2}\right)\right)\hat{u}^n(\xi).$$

The above equation has the following form

$$u_j^{n+1}(\xi)=g(\xi)\hat{u}^n(\xi),$$

where $g(\xi)$ is called the amplification factor. For stability, we need that $|g(\xi)|=1+\alpha\Delta t$ for all ξ. Then we have

$$||\hat{u}^{n+1}||_2\leq(1+\alpha\Delta t)||\hat{u}^n||_2$$

and then

$$||u^{n+1}||_2=(1+\alpha\Delta t)||u^n||_2.$$

Forward Euler for the diffusion equation is stable if

$$|g(\xi)|=\left|1-\frac{4b\Delta t}{h^2}\sin^2\left(\frac{\xi h}{2}\right)\right|\leq 1.$$

This holds when

$$\begin{aligned}-1\leq 1-\frac{4b\Delta t}{h^2}\sin^2\left(\frac{\xi h}{2}\right)&\leq 1\\ -2\leq -\frac{4b\Delta t}{h^2}\sin^2\left(\frac{\xi h}{2}\right)&\leq 0\\ \Delta t\leq\frac{2}{\frac{4b}{h^2}\sin^2(\frac{\xi h}{2})}&=\frac{h^2}{2b\sin^2(\frac{\xi h}{2})}\end{aligned}$$

The most severe restriction occurs when $\sin^2(\xi h/2)=1$, and so for stability we require that

$$\Delta t\leq\frac{h^2}{2b}.$$

To perform Von Neumann Analysis you need not go through all of the formalism of the previous example. In general, you assume solution of the form

$$u_j^n=e^{i\xi x_j}\text{ and }u_j^{n+1}=g(\xi)e^{i\xi x_j}.$$

Plug this into the difference equation and calculate $g(\xi)$. Looking at Fourier modes is enough to tell you about the stability of the solution. More generally, if you have a multilevel scheme, assume a solution of the form

$$u_j^n=g(\xi)^n e^{ix_j\xi}.$$

Example: Leap-Frog (3-level scheme, centered difference in time) is unstable for diffusion equation by von Neumann Analysis. The difference equation is

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = b\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2}.$$

Computing the amplification factor,

$$\frac{g^{n+1}(\xi) - g^{n-1}(\xi)}{2\Delta t} = \frac{-4b}{h^2} g^n(\xi)\sin^2\left(\frac{\xi h}{2}\right).$$

Which we can then simplify to obtain a quadratic equation for g,

$$g^2 + \frac{8\Delta t b}{h^2}\sin^2\left(\frac{\xi h}{2}\right) g - 1 = 0.$$

The solutions are

$$g_{+/-} = \frac{-4\Delta t b}{h^2}\sin^2\left(\frac{\xi h}{2}\right) \pm \left(\left(\frac{4\Delta t b}{h^2}\sin^2\left(\frac{\xi h}{2}\right)\right)^2 + 1\right)^{1/2}$$

Note that $|g_-| > 1$ for some ξ and thus the scheme is unstable.