Math 228B
Week 4 Notes
Tuesday, 01/27/09

1 Von Neumann Analysis Cont.

Notice that the Von Neumann analysis we saw last time technically only applies to constant coefficient problems on periodic domains. But what if we went ahead and tried to apply it to other problems anyways?

Consider the variable coefficient diffusion problem (in conservation form)

$$u_t = (b(x)u_x)_x, \quad b > 0,$$

which models a situation where the "flux" varies spacially. We *could* apply the chain rule to the right hand side and discretize as we have previously, but perhaps we should discretize the problem with the conservation phenomenon in mind.

Consider the spacial grid below with "nodes" at $x_j = hj$ and "edges" at $x_{j+1/2}$, and define $J = -b(x)u_x$ (i.e., the flux).

Place Picture here

Applying a centered difference scheme to calculate $-J_x$ we have

$$-J_x \approx -\left(\frac{J_{j+1/2} - J_{j-1/2}}{h}\right).$$

But, by applying a centered difference scheme to the definition of J, we have

$$-J_x \approx -\left(\frac{b_{j+1/2}\left(\frac{u_{j+1}-u_j}{h}\right) + b_{j-1/2}\left(\frac{u_j-u_{j-1}}{h}\right)}{h}\right).$$

Simplifying this expression yields

$$-J_x \approx \frac{b_{j-1/2}u_{j-1} - (b_{j-1/2} + b_{j+1/2})u_j + b_{j-1/2}u_{j+1}}{h^2}.$$

Notice that if b is a constant, then this expression would reduce to the standard discrete laplacian we are familiar with.

Now what if we wanted to apply Forward-Euler to this spacial discretization?

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{b_{j-1/2}u_{j-1} - (b_{j-1/2} + b_{j+1/2})u_j + b_{j-1/2}u_{j+1}}{h^2} \tag{1}$$

If $b(x)$ were sufficiently smooth, we would expect to the b_k's to be approximately equal. And we know that applying Von Neumann analysis to the problem $u_t = \tilde{b}u_{xx}$ leads us to the stability restriction

$$\Delta t \leq \frac{h^2}{2\tilde{b}} \tag{2}$$

So what if we just chose $\tilde{b} = \max_{x\in\Omega}(b(x))$? Would this be reasonable?

Again, consider equation (1). It has the form $u^{n+1} = (I + \Delta t B)u^n$ where B is a matrix of variable coefficients. Calculating the ∞-norm of this operator yields

$$\|I + \Delta t B\|_\infty = \max_j \left(\frac{\Delta t}{h^2} b_{j-1/2} + |1 - \frac{\Delta t}{h^2}(b_{j-1/2} + b_{j+1/2})| + \frac{\Delta t}{h^2} b_{j+1/2} \right).$$

So if we choose Δt small enough, the middle term is guranteed to be positive, and we can drop the absolute values. In this case, the above expression reduces to $\|I + \Delta t B\|_\infty = 1$, and the scheme is stable. More formally, this condition on the size of Δt is equivalent to

$$\Delta t \leq \frac{h^2}{b_{j-1/2} + b_{j+1/2}} \quad \text{for all} \quad j. \tag{3}$$

Notice that (2) is actually a stronger condition than (3), so it will imply stability of the scheme. Furthermore, it is easier to derive and more practical.

In a similar fashion, we can extend Von Neumann analysis (which tells us that Crank-Nicolson and Backward-Euler are unconditionally stable for the constant coefficient heat equation) to show that C-N and B-E are stable for the variable coefficient problem.

2 Higher Dimensions

What if we wish to use an implicit scheme, such as C-N, to solve

$$u_t = b\Delta u$$

in higher spacial dimensions? To do this, we must solve the system

$$(I - \frac{b\Delta t}{2}L)u^{n+1} = (I + \frac{b\Delta t}{2}L)u^n$$

for each time step. If we define $A = (I - \frac{b\Delta t}{2}L)$, do we know that A is an invertible operator? Actually, yes! Clearly A has the same eigenvectors as L, and we know from last quarter that L has negative eigenvalues, and thus, A is positive definite and non-singular. However, inverting this operator can be very computationally intensive in higher dimensions, since we lose the tri-diagonal structure that it has in one spacial dimension.

Perhaps, for higher dimensions it would be more efficient to use an explicit scheme like Forward-Euler. This turns out to be untrue, as the stability restrictions on Δt become more severe. In two dimensions, we have $\Delta t \leq \frac{h^2}{4b}$. In three dimensions this becomes $\Delta t \leq \frac{h^2}{6b}$, and so on ... So in general, we will wish to use an implicit scheme.

Question. How do we solve $Au^{n+1} = r$ efficiently?

Answer (Bad!)**.** Normal Gaussian elimination takes $\mathcal{O}(M^2)$ work, where M is the total number of spacial grid points.

Answer (Better)**.** Iterative methods? Multi-Grid, SOR, and Conjugate Gradient all worked well for Poisson equation. Will they work well to invert our new (slightly different) operator? How efficient are they?

- C-G will obviously work because A is a symmetric, positive definite operator.

- MG will also work, because A has the same eigenvectors as L. Therefore we can still use the method of smoothing and coarse-grid correction.
- SOR will also work, but we will need to choose a different ω_{opt} (this was actually a homework problem last semester).

Question. Since the efficiency of these methods is related to the condition number of our operator, what is the condition number of A?

Answer. If L is the standard discrete Laplacian, then the condition number

$$K(L) = \frac{4/h^2}{1} = \mathcal{O}(h^{-2}).$$

So it is easy to see from the form of A that its condition number is

$$K(A) = \frac{1 + \mathcal{O}(\Delta t/h^2)}{\mathcal{O}(1)} = \mathcal{O}(\Delta t/h^2).$$

So, assuming that Δt is small, we have a well conditioned system to solve.