Lectures, Day 3/5

(Here is a figure)

$$u_j = \frac{1}{h}\int_{C_j} u(x)dx = u(x_j) + O(h^2)$$

$$u_t + (f(u))_x = 0$$

The integral conservation law on C_j

$$\frac{d}{dt}\int_{C_j} u(x,t)dx = f(u(x_{j-1/2},t)) - f(u(x_{j+1/2},t))$$

Integrate this equation over $t \in [t_n, t_{n+1}]$

$$\frac{1}{h}\int_{C_j} u(x,t_{n+1})dt - \frac{1}{h}\int_{C_j} u(x,t_n)dx = \frac{1}{h}\int_{t_n}^{t_{n+1}} f(u(x_{j-1/2},t))dt - \frac{1}{h}\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2},t))dt$$

$$u_j^{n+1} = u_j^n - \frac{1}{h}\left(\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2},t))dt - \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2},t))dt\right)$$

Let $F_{j-1/2}^n \sim \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{j-1/2},t))dt$
This gives a numerical method

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{h}(F_{j+1/2}^n - F_{j-1/2}^n) \rightarrow \text{discrete conservation law}$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{F_{j+1/2}^n - F_{j-1/2}^n}{h} = 0 \rightarrow \text{discritezed version of the differential form}$$

This form ensures discrete consrevation.
(Here is a figure)

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{h}(F_{j+1/2}^n - F_{j-1/2}^n)$$

$$u_{j-1}^{n+1} = u_j^n - \frac{\Delta t}{h}(F_{j-1/2}^n - F_{j-3/2}^n)$$

Upwinding for linear advection $u_t + au_x = 0$

$$a > 0$$

$$f(u) = au$$

Upwinding flux is $F_{j-1/2}^n = au_{j-1}^n$
Lax-Wendroff flux
two step LW

$$u_{j+1/2}^{n+1/2} = \frac{1}{2}(u_j^n + u_{j+1}^n) - \frac{\Delta t}{2h}(f(u_{j+1}^n) - f(u_j^n))$$

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{h}(f(u_{j+1/2}^{n+1/2}) - f(u_{j-1/2}^{n-1/2}))$$

For linear advection

$$F_{j-1/2}^{\text{LW}} = \frac{a}{2}(u_{j-1}^n + u_j^n) - \frac{a^2\Delta t}{2h}(u_j^n - u_{j-1}^n)$$

add & subtract $au_{j-1}^n = F_{j-1/2}^{\text{UP}}$

$$F^{\text{LW}}_{j-1/2} = au^n_{j-1} + \frac{a}{2}(u^n_j - u^n_{j-1}) - \frac{a^2\Delta t}{2h}(u^n_j - u^n_{j-1}) = au^n_{j-1} + \frac{a}{2}(1 - \frac{a\Delta t}{h})(u^n_j - u^n_{j-1})$$

Upwind flux, Second order correction to the upwind flux

Idea of high-resolution methods

$$F_{j-1/2} = F^{\text{UP}}_{j-1/2} + (F^{\text{LW}}_{j-1/2} - F^{\text{UP}}_{j-1/2})\phi$$

ϕ is called a lux limiter funciton. It depends on the solution.

Want $\phi \to 1$ when the solution is smooth.
$\phi \to 0$ near shapr transitions.

Godunov's method (REA method)

Start with u^n_j ... averages over the cells Reconstruct ... from cell averages, find some function with those averages. e.g. PW constatnt, PW linear

1. (Here is a figure)

Evolve $\to$ Solve the PDE for time length Δt with the reconstruction as initial data.

2. (Here is a figure)

Average $\to$ over each cell to find u^{n+1}

PW constant reconstruction reduces to upwinding for linear advection.

This is the way to generalize upwinding for nonlinear problem.

$\to$ involves solving Riemann problems at the cell edges.

Riemann problem

$$u_t + f(u) = 0$$

with initial data

$$u(x, 0) = u_l x < 0, u_r x > 0$$

How to get higher accuracy?

PW linear reconstruction

$$\bar{u}(x, t_n) = u^n_j + \sigma^n_j (x - x_j) x \in [x_{j-1/2}, x_{j+1/2}]$$

What slope to choose?

$\sigma = 0$ ... Godunov's method

centered $\sigma_j = \frac{u_{j+1} - u_j}{2h}$ Fromm's scheme

upwind $(a > 0)$ $\sigma_j = \frac{u_j - u_{j-1}}{h}$ Beam-Warming

downwind $(a > 0)$ $\sigma_j = \frac{u_{j+1} - u_j}{h}$ Lax-Wendroff

(Here is a figure)