Lecture of Day 3/11

Higher order REA method: Piecewise constant reconstruction: Godunov's method for advection this is upwinding. Try piecewise linear reconstruction.

$$\tilde{u}(x, t_n) = u_j^n + \sigma_j^n (x - x_j) \ , \quad x_{j-\frac{1}{2}} \le x \le x_{j+\frac{1}{2}}.$$

$\tilde{u}^n$ is conservative for any slope σ_j^n:

- Downwind slope: Lax-Wendroff
- Upwind slope: Bean-Warming
- Centered slope: Fromm's scheme.

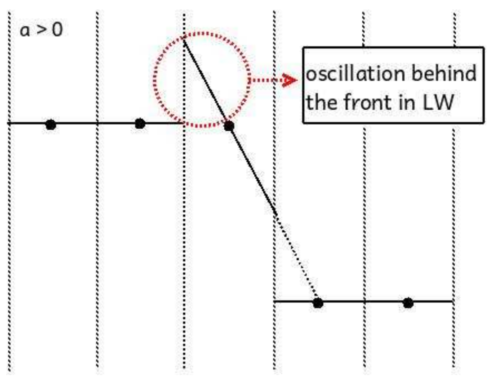

Figure 1: An overshoot occurs behind the front by LW ($a > 0$).

What to do to avoid oscillation? Use a slope limiter. e.g. *minmod slope*:

$$\sigma_j^n = \text{minmod}\left(\frac{u_j - u_{j-1}}{h}, \frac{u_{j+1} - u_j}{h}\right)$$

where

$$\text{minmod}(a, b) = \begin{cases} a & \text{if } |a| \le |b| \text{ and } ab > 0 \\ a & \text{if } |b| \le |a| \text{ and } ab > 0 \\ 0 & \text{if } ab < 0 \end{cases}$$

Generally this is still very diffusive. A better choice is *monotonized centered difference* (MC slope):

$$\sigma_j^n = \text{minmod}\left(\frac{u_{j+1} - u_{j-1}}{2h}, \frac{2(u_{j+1} - u_j)}{h}, \frac{2(u_j - u_{j-1})}{h}\right).$$

For three or more arguments: if the minmod have the same sign, then choose the one with smallest absolute value; if they are not of the same sign, choose 0.

Let's go back to the discrete conservation law:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{h}(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n),$$

$$F_{j-\frac{1}{2}}^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{j-\frac{1}{2}}, t))\underset{.}{t}$$

For advection equation,

$$f(u) = au \ , \quad F_{j-\frac{1}{2}}^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} au(x_{j-\frac{1}{2}}, t)\underset{.}{t}.$$

Assume $a > 0$,

$$F^n_{j-\frac{1}{2}} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} au(x_{j-\frac{1}{2}} - a(t-t_n), t_n)\text{ţ}.$$

Assume piecewise linear reconstruction,

$$\begin{aligned} F^n_{j-\frac{1}{2}} &= \frac{a}{\Delta t}\int_{t_n}^{t_{n+1}} u^n_{j-1} + \sigma^n_{j-1}(x_{j-\frac{1}{2}} - a(t-t_n) - x_{j-1})\text{ţ} \\ &= au^n_{j-1} + \frac{ah}{2}\left(1 - a\frac{\Delta t}{h}\right)\sigma^n_{j-1} \end{aligned}$$

where the right hand terms au^n_{j-1} is the upwinding and $\frac{ah}{2}\left(1 - a\frac{\Delta t}{h}\right)\sigma^n_{j-1}$ the second order correction. Notice that $h\sigma^n_{j-1} = u_j - u_{j-1}$ in LW. More generally,

$$F^n_{j-\frac{1}{2}} = \begin{cases} au^n_{j-1} + \frac{ah}{2}\left(1 - a\frac{\Delta t}{h}\right)\sigma^n_{j-1} & a \geq 0 \\ au^n_j - \frac{ah}{2}\left(1 + a\frac{\Delta t}{h}\right)\sigma^n_j & a < 0 \end{cases}$$

In general,

$$F_{j-\frac{1}{2}} = F^{\text{UP}}_{j-\frac{1}{2}} + \left|\frac{a}{2}\right|\left(1 - \left|\frac{a\Delta t}{h}\right|\right)\delta^n_{j-\frac{1}{2}}$$

where $\delta^n_{j-\frac{1}{2}}$ is a limited version of the difference $(\Delta u)_{j-\frac{1}{2}} = u_j - u_{j-1}$.

How to measure smoothness in a grid function (using only cell averages)? Define a ratio

$$\theta_{j-\frac{1}{2}} = \frac{\Delta u_{J_{\text{UP}}-\frac{1}{2}}}{\Delta u_{j-\frac{1}{2}}}\ , \quad \text{where } J_{\text{UP}} = \begin{cases} j-1 & \text{if } a > 0 \\ j+1 & \text{if } a < 0 \end{cases}$$

For smooth function, we expect $\theta = 1 + \mathcal{O}(h)$ except near extrema. θ far from one indicates a lack of smoothness.

$$\delta^n_{j-\frac{1}{2}} = \phi(\theta^n_{j-\frac{1}{2}})\Delta u_{j-\frac{1}{2}}$$

where ϕ is called the *flux limiter function*. Note that

- $\phi = 0$: Upwinding
- $\phi = 1$: Lax-Wendroff
- $\phi = \theta$: Beam-Warming.

Range of ϕ will be $[0, 2]$:

- $\phi < 1$: more limiting - smoothing
- $\phi > 1$: limiter gives a steeper reconstruction than LW.

Some high-resolution limiters:

- minmod: $\phi(\theta) = \text{minmod}(1, \theta)$
- superbee: $\phi(\theta) = \max(0, \min(1, 2\theta), \min(2, \theta))$
- MC: $\phi(\theta) = \max\left(0, \min(\frac{1+2\theta}{2}, 2, 2\theta)\right)$
- van Leen: $\phi(\theta) = \frac{\theta + |\theta|}{1 + |\theta|}$

We want to avoid introducing unphysical oscillation in the numerical solution. How do we measure oscillation? *Total variation* of the grid function is

$$\mathrm{TV}(u) = \sum_j |u_j - u_{j-1}| \,.$$

For f a differentiable function on $[a, b]$:

$$\mathrm{TV}(f) = \int_a^b |f'(x)| \,\underset{.}{\mathrm{x}}.$$

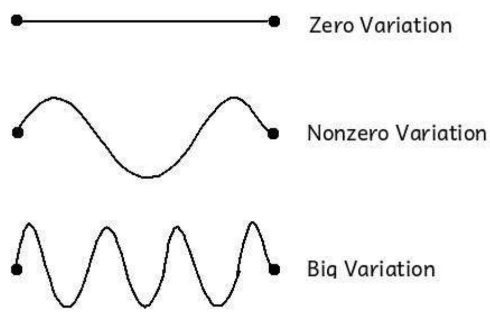

Figure 2: Total variation: a quantity that measures the oscillation.

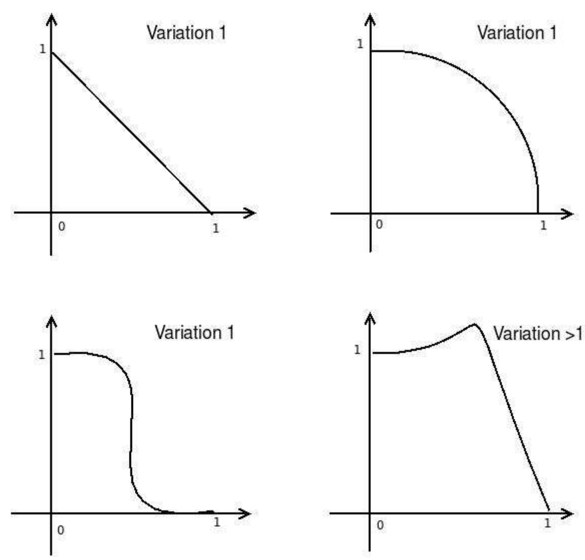

Figure 3: Increase the variation if the function is not monotone.