

Lectures, Day 3/12

$$\text{TV}(u^n) = \sum_j |u_j^n - u_{j-1}^n|.$$

The solution to the advection equation has constant variation. Avoid oscillations by avoiding growth in variation.

[Handout: Numerical solution by UP, LW, Minmod]

A two scheme is called *total variation diminishing* (TVD) if

$$\text{TV}(u^{n+1}) \leq \text{TV}(u^n).$$

From HW3, upwind is TVD but Lax-Wendroff is not.

How to choose the limiter $\phi(\theta)$ to give a second-order scheme for smooth functions that is TVD (no dispersive oscillation)? Remind that

$$F = F^{\text{UP}} + (F^{\text{LW}} - F^{\text{UP}})\phi(\theta).$$

For second order accuracy we need $\phi(1) = 1$. Need ϕ to Lipschitz continuous (bounded left or right derivatives, smooth at $\theta = 1$.)

Write out the scheme for $a > 0$:

$$u_j^{n+1} = u_j^n - \nu(u_j^n - u_{j-1}^n) - \frac{\nu(1-\nu)}{2} \left(\phi(\theta_{j+\frac{1}{2}})(u_{j+1}^n - u_j^n) - \phi(\theta_{j-\frac{1}{2}})(u_j^n - u_{j-1}^n) \right).$$

A scheme of the form

$$u_j^{n+1} = u_j^n - C_{j-1}^n(u_j^n - u_{j-1}^n) + D_j^n(u_{j+1}^n - u_j^n)$$

is TVD if

$$C_{j-1}^n \geq 0, \quad D_j^n \geq 0, \quad \text{and} \quad C_j^n + D_j^n \leq 1.$$

Try

$$C_{j-1}^n = \nu - \frac{\nu(1-\nu)}{2} \phi(\theta_{j-\frac{1}{2}})$$

$$D_j^n = -\frac{\nu(1-\nu)}{2} \phi(\theta_{j+\frac{1}{2}}) \leq 0 \quad \text{for } 0 < \nu < 1 \text{ and } \phi \approx 1$$

Trick to get TVD is to write, by the definition of $\theta_{j+\frac{1}{2}}$ with $a > 0$,

$$u_{j+1}^n - u_j^n = \frac{u_j^n - u_{j-1}^n}{\theta_{j+\frac{1}{2}}}.$$

Substituting this we rewrite the scheme, which yields

$$C_{j-1}^n = \nu + \frac{\nu(1-\nu)}{2} \left(\frac{\phi(\theta_{j+\frac{1}{2}})}{\theta_{j+\frac{1}{2}}} - \phi(\theta_{j-\frac{1}{2}}) \right)$$

$$D_j^n = 0$$

For TVD we need $0 \leq C_{j-1}^n \leq 1$, and still $0 \leq \nu \leq 1$ for CFL. After some algebraic rearrangement we have that the scheme is TVD if

$$\left| \frac{\phi(\theta_1)}{\theta_1} - \phi(\theta_2) \right| \leq 2 \quad \text{for all } \theta_1, \theta_2.$$

That is, requiring

$$0 \leq \frac{\phi(\theta)}{\theta} \leq 2, \quad 0 \leq \phi(\theta) \leq 2, \quad \text{for all } \theta > 0,$$

and $\theta(\theta) = 0$ for $\theta \leq 0$. To conclude, A scheme is TVD if

$$0 \leq \phi(\theta) \leq \min\text{mod}(2, 2\theta).$$

[Handout: TVD region and limiters]

These four limiters satisfy a symmetric condition

$$\phi\left(\frac{1}{\theta}\right) = \frac{\phi(\theta)}{\theta}.$$

If the function is symmetric, the reconstruction is symmetric.