MAT 228B

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Lecture 1

Introduction

Last quarter, we covered the following material:

- Poisson equation: $\Delta u = f$

  In particular we used the Poisson equation as a model problem to learn

  - how to discretize the Poisson operator
  - how to solve the Possion equation
  - the accuracy and convergence of the aforementioned methods

This quarter, we'll be tackling:

- time dependent problems such as

$$\begin{array}{ll} u_t = D\Delta u & \text{diffusion/heat equation} \\ u_t + au_x = 0 & \text{advection equation} \\ u_tt = c^2\Delta u & \text{wave equation} \end{array}$$

- mixed equations

$$u_t + au_x = Du_{xx} + R(u) \qquad \text{advection-diffusion-reaction equation}$$

- non-linear equations such as

$$u_t + uu_x = 0 \quad \text{OR} \quad u_t + uu_x = \epsilon u_{xx} \qquad \text{Burger's equation}$$

  The latter non-linear example of Burger's equation is also an example of a singular perturbation.

- conservation laws

$$u_t + (f(u))_x = 0, \text{ where } f \text{ is the flux function}$$

Classification of PDEs

Recall that the Poisson equation is an elliptic equation. More generally, we can have $Lu = f$, where

$$L = \sum_{ij} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_j b_j(x) \frac{\partial}{\partial x_j} + c(x).$$

L is elliptic if the matrix a_{ij} is positive or negative definite. Moreover, if L is an elliptic operator, then $u_t = Lu + f$ is a parabolic equation.

Examples of hyperbolic equations are the advection and wave equations previously mentioned. Namely, the first order system $\vec{u}_t + A\vec{u}_x = 0$ is hyperbolic if A has real eigenvalues and is diagonalizable.

To see how $u_t = c^2 \Delta u$ is hyperbolic, let

$$p = u_t \qquad \text{and} \qquad q = -u_x$$

which gives

$$p_t = u_{tt} = c^2 u_{xx} = -c^2 q_x \qquad \text{and} \qquad q_t = -u_{xt} = -u_{tx} = -p_x$$

Thus leaving us with

$$\begin{pmatrix} p \\ q \end{pmatrix}_t = \begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}_x \implies \begin{pmatrix} p \\ q \end{pmatrix}_t + \begin{pmatrix} 0 & c^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}_x = 0$$

Note that the eigenvalues of this operator are $\pm c$, satisfying the criteria of a hyperbolic equation.

Conservation Laws

Let $\rho(x,t)$ be a density (e.g. $\frac{\text{mass}}{\text{length}}$ or $\frac{\text{moles}}{\text{length}}$).

Let F be a flux function (e.g. $\frac{\text{mass}}{\text{time}}$).

Suppose $F = F(\rho(x,t))$, or more generally $F = F(x, \rho, \rho_x)$.

Then the total amount of stuff in the interval $[x_1, x_2]$ is

$$M = \int_{x_1}^{x_2} \rho(x,t)dx \qquad \text{and} \qquad \frac{dM}{dt} = \frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)dx = F(\rho(x_1,t)) - F(\rho(x_2,t))$$

Invoking the Fundamental Theorem of Calculus gives us

$$\frac{dM}{dt} = \frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)dx = \int_{x_1}^{x_2} -(F(\rho(x,t)))_x dx \implies \int_{x_1}^{x_2} \rho_t + (F(\rho))_x dx = 0$$

The latter form is the integral/weak form of the conservation law. Because x_1 and x_2 are arbitrary and ρ is smooth enough, we also get the differential form of the conservation law

$$\rho_t + (F(\rho))_x = 0$$

Examples:

1) Let u be the concentration of a chemical

$$\implies u_t + (F(u))_x = 0$$

2) Assume $F = au$ (flux is proportional to the concentration)

$$\implies u_t + au_x = 0 \leftarrow \text{ advection equation}$$

3) Now assume $F = -Du_x$ (flux is proportional to the negative of the gradient)

$$\implies u_t + (-Du_x)_x = 0 \implies u_t = (Du_x)_x \leftarrow \text{ diffusion equation}$$

Near the end of the quarter we will study nonlinear hyperbolic conservation laws (e.g. Euler equation (gas dynamics)):

$$\begin{array}{ll} \rho_t + (v\rho)_x = 0 & \text{conservation of mass} \\ (\rho v)_t + (\rho v^2 + p)_x = 0 & \text{conservation of linear momentum} \\ E_t + (v(E+p))_x = 0 & \text{conservation of energy} \end{array}$$

Advection vs Diffusion

Compare the solutions of

$$u_t + au_{xx} \text{ and } u_t = Du_x \text{ on } \mathbb{R}$$

We need an initial condition $u(x,0) = u_0(x)$. The solution to the advection equation is

$$\begin{array}{l} u(x,t) = u_0(x-at) \leftarrow \text{ initial data translates at a constant speed} \\ u_t(x,t) = u_0(x-at)(-a) = -au_x \end{array}$$

(picture to be inserted)

For the heat equation (diffusion equation), $u_t = Du_{xx}$, we look at a special case:

$$u_0(x) = e^{i\xi x} \text{ for which we make the guess for the solution of the form } u(x,t) = g(t)e^{i\xi x}$$

giving us

$$g'(t)e^{i\xi x} = -D\xi^2 g(t)e^{i\xi x} \implies g(t) = e^{-D\xi^2 t}$$

Thus our solution is $u(x,t) = e^{-D\xi^2 t}e^{i\xi x} \implies |u(x,t)| = e^{-D\xi^2 t}$.

The amplitude decays and high wave numbers ξ decay more rapidly than low ξ, which gives us smoother solutions as time evolves.

In fact, if the initial data is discontinuous, the solution is still C^∞ for all $t > 0$. In particular, if the initial data is in L^2, then the solution is in C^∞ for all $t > 0$.