Math 228B
Homework 1
Due Tuesday, 02/03/09.

1. Consider the following PDE.

$$\begin{gathered} u_t = 0.01\, u_{xx} + 1 - \exp(-t), \quad 0 < x < 1 \\ u(0,t) = 0 \quad u(1,t) = 0 \\ u(x,0) = 0 \end{gathered}$$

   (a) Write a program to solve the problem using Crank-Nicolson up to time $t = 1$, and perform a refinement study that demonstrates that the method is second-order accurate in space and time.

   (b) Solve the problem using a forward Euler method up to time $t = 1$. Demonstrate in a refinement study that the method is first-order in time and second-order in space.

2.

$$u_t = u_{xx}, \quad 0 < x < 1$$

$$u(0,t) = 1, \quad u(1,t) = 0$$

$$u(x,0) = \begin{cases} 1 & \text{if } x < 0.5 \\ 0 & \text{if } x \geq 0.5 \end{cases}$$

   (a) Use Crank-Nicolson with grid spacing $h = 0.02$ and time step 0.1 to solve the problem up to time $t = 1$. Comment on your results. What is wrong with this solution?

   (b) Give a mathematical argument to explain the unphysical behavior you observed in the numerical solution.

   (c) Experiment with smaller time steps. How small does the time step need to be to get reasonable results?

   (d) What happens to the numerical solution as $\Delta t \to 0$ with the ratio $\Delta t/h$ fixed? Explain. Would this same behavior occur using backward Euler in place of Crank-Nicolson? Explain.

3. Derive a stability restriction on the time step for solving the diffusion equation using the second-order accurate explicit Runge-Kutta method

$$\begin{aligned} y^* &= y^n + \Delta t f(y^n) \\ y^{n+1} &= y^n + \frac{\Delta t}{2}\big(f(y^n) + f(y^*)\big), \end{aligned}$$

   for time stepping. This scheme is second-order accurate in space and time, and the error is of the form $\mathcal{O}(h^2) + \mathcal{O}(\Delta t^2)$. Assuming that the constants hidden in the big-oh terms are about the same for the spatial and temporal errors, explain why this scheme would not be used in place of forward Euler.

4. Consider the forward time, centered space discretization

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2h} = b\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2},$$

to the convection-diffusion equation,

$$u_t + au_x = bu_{xx}, \quad b > 0.$$

(a) Let $\nu = a\Delta t/h$ and $\mu = b\Delta t/h^2$. Use von Neumann analysis to show that the scheme is stable if $\mu \leq 1/2$.

(b) Let $a = 80$, $b = 1$, $h = 0.05$. Generate a numerical solution on the spatial domain $[0, 1]$ with periodic boundary conditions using $\Delta t = 0.25h^2/b$ with initial condition $u(x, 0) = \exp(-20(x - 0.5)^2)$. What happens? Does your stability analysis predict this?

(c) Since the solution to the PDE does not grow in time, it seems reasonable to require that the numerical solution not grow in time. Show that the numerical solution does not grow (in 2-norm) if and only if $\nu^2 \leq 2\mu \leq 1$. This is called strict or practical stability, and as the name suggests it is the restriction one would use in practice.

(d) Generate a numerical solution up to time $t = 10^{-2}$.