Math 228B
Homework 2
Due Thursday, 2/19/09.

1. In class, we showed that the two-dimensional Peaceman-Rachford ADI scheme is unconditionally stable and second-order accurate in time. This can be thought of as either an approximate factorization or as a fractional step method. By adapting the fractional step idea to three-dimensions we get the scheme

$$\left(I - \frac{b\Delta t}{3}L_x\right)u^* = \left(I + \frac{b\Delta t}{3}L_y + \frac{b\Delta t}{3}L_z\right)u^n$$
$$\left(I - \frac{b\Delta t}{3}L_y\right)u^{**} = \left(I + \frac{b\Delta t}{3}L_x + \frac{b\Delta t}{3}L_z\right)u^*$$
$$\left(I - \frac{b\Delta t}{3}L_z\right)u^{n+1} = \left(I + \frac{b\Delta t}{3}L_x + \frac{b\Delta t}{3}L_y\right)u^{**}.$$

   (a) Use von Neumann analysis to show that this scheme is conditionally stable. This is an example of how certain desirable properties of a numerical scheme can be lost when using fractional stepping.

   (b) What temporal accuracy do you expect from this scheme? Explain.

2. Consider

$$u_t = 0.1\,\Delta u \text{ on } \Omega = (0,1)\times(0,1)$$
$$\frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \partial\Omega$$
$$u(x,y,0) = \exp\left(-10((x-0.4)^2 + (y-0.4)^2)\right)$$

   (a) Write a program to solve this PDE using the Peaceman-Rachford ADI scheme on a cell-centered grid. Use a direct solver for the tridiagonal systems. Recall that in a cell-centered discretization the solution is stored at the grid points $(x_i, y_j) = (h(i-0.5), h(j-0.5))$ for $i, j = 1 \ldots N$ and $h = 1/N$ (see Homework 1 from 228A). This discretization is natural for handling Neumann boundary conditions, and it is often used to discretize conservation laws. At the grid points adjacent to the boundary, the one-dimensional discrete Laplacian for homogeneous Neumann boundary conditions is

$$u_{xx}(x_1) \approx \frac{-u_1 + u_2}{h^2}.$$

   See Homework 1 from 228A to see how to derive this by flux differencing or by using ghost cells.

   (b) Perform a refinement study to show that your numerical solution is second-order accurate in space and time (refine time and space simultaneously using $\Delta t = h$) at time $t = 1$.

   (c) Show that the spatial integral of the solution to the PDE does not change in time. That is

$$\frac{d}{dt}\int_\Omega u\,dV = 0.$$

(d) Prove that the solution to the discrete equations satisfies the discrete conservation property

$$\sum_{i,j} u_{i,j}^n = \sum_{i,j} u_{i,j}^0$$

for all n. Demonstrate this property with your code.

3. The FitzHugh-Nagumo equations

$$\frac{\partial v}{\partial t} = D\Delta v + (a - v)(v - 1)v - w + I$$
$$\frac{\partial w}{\partial t} = \epsilon(v - \gamma w).$$

are used in electrophysiology to model the cross membrane electrical potential (voltage) in cardiac tissue and in neurons. Assuming that the spatial coupling is local and passive results the term which looks like the diffusion of voltage. The state variables are the voltage v and the recovery variable w.

(a) Write a program to solve the FitzHugh-Nagumo equations on the unit square with homogeneous Neumann boundary conditions for v (meaning electrically insulated). Use a fractional step method to handle the diffusion and reactions separately. Use an ADI method for the diffusion solve. Describe what ODE solver you used for the reactions and what fractional stepping your chose.

(b) Use the following parameters $a = 0.1$, $\gamma = 2$, $\epsilon = 0.005$, $I = 0$, $D = 5 \cdot 10^{-5}$, for $h = 0.01$ and initial conditions

$$v(x, y, 0) = \exp\left(-100(x^2 + y^2)\right)$$
$$w(x, y, 0) = 0.0.$$

Note that $v = 0$, $w = 0$ is a stable steady state of the system. Call this the rest state. For these initial conditions the voltage has been raised above rest in the bottom corner of the domain. Generate a numerical solution up to time $t = 300$. What time step did you use and why? Visualize the voltage and describe the solution.

(c) Use the same parameters from part (b), but use the initial conditions

$$v(x, y, 0) = 1 - 2x$$
$$w(x, y, 0) = 0.05y,$$

and run the simulation until time $t = 600$. Show the voltage at several points in time (pseudocolor plot, or contour plot, or surface plot $z = V(x, y, t)$) and describe the solution.

The dynamics of excitable media is a fascinating subject from both the mathematical and physiological perspectives. The electrical patterns that you simulated in part (c) are related to cardiac arrhythmias. For more information see the book *Mathematical Physiology* by Keener and Sneyd.

Just for fun, (there is no need to turn this in or even do it) try to find an input current $I(x, y, t)$ in the form of a short pulse (e.g. $I(x, y, t) = f(x, y) \exp\left(-\kappa(t - t_p^2)\right)$) so that the normal electrical wave from part (b) degenerates into an arrhythmia like that from part (c). Then try to find a pulse of current that will eliminate the arrhythmia. This second task may be easier. What do the doctors on TV do?