Math 228B
Homework 3
Due Friday, 3/06/09.

1. Write programs to solve the advection equation

$$u_t + au_x = 0,$$

on $[0, 1]$ with periodic boundary conditions using upwinding and Lax-Wendroff. For smooth solutions we expect upwinding to be first-order accurate and Lax-Wendroff to be second-order accurate, but it is not clear what accuracy to expect for nonsmooth solutions.

   (a) Let $a = 1$ and solve the problem up to time $t = 1$. Perform a refinement study for both upwinding and Lax-Wendroff with $\Delta t = 0.8h$ with a smooth initial condition. Compute the rate of convergence in the 1-norm, 2-norm, and max-norm. Note that the exact solution at time $t = 1$ is the initial condition, and so computing the error is easy.

   (b) Repeat the previous problem with the discontinuous initial condition

$$u(x, 0) = \begin{cases} 1 & \text{if } |x - 1/2| < 1/4 \\ 0 & \text{otherwise} \end{cases}$$

2. Consider three-point explicit schemes for the linear advection equation in the real line of the form

$$u_j^{n+1} = u_j^n - C\left(u_j^n - u_{j-1}^n\right) + D\left(u_{j+1}^n - u_j^n\right).$$

Show that

$$\sum_j \left|u_j^{n+1} - u_{j-1}^{n+1}\right| \le \sum_j \left|u_j^n - u_{j-1}^n\right| \qquad (1)$$

if $C \ge 0$, $D \ge 0$, and $C + D \le 1$. When the numerical solution of a scheme satisfies (1) the scheme is total variation diminishing or TVD. Put upwinding and Lax-Wendroff into the above form, and show that upwinding is TVD when it is stable and that Lax-Wendroff is not TVD. Give an interpretation for the meaning of TVD and explain how this relates to the numerical solutions from problem 1b.

3. For solving the heat equation we frequently use Crank-Nicolson. This scheme is not often used for hyperbolic problems. For the linear advection equation, Crank-Nicolson is

$$u_j^{n+1} - u_j^n + \frac{\nu}{4}(u_{j+1}^n - u_{j-1}^n) + \frac{\nu}{4}(u_{j+1}^{n+1} - u_{j-1}^{n+1}) = 0.$$

   (a) Show that Crank-Nicolson is unconditionally stable for the advection equation.

   (b) Use von Neumann analysis to show that for a periodic domain $\|u^n\|_2 = \|u^0\|_2$ for all n. This scheme is said to nondissipative. This seems reasonable because this is a property of the PDE.

   (c) Solve the advection equation on the periodic domain $[0, 1]$ with the initial condition from problem 1b. Show the solution and comment on your results.