Math 228B
Homework 4
Due Friday, 3/20/09.

1. In one spatial dimension the linearized equations of acoustics (sound waves) are

$$\begin{aligned} p_t + K u_x &= 0 \\ \rho u_t + p_x &= 0, \end{aligned}$$

   where u is the velocity and p is the pressure, ρ is the density, and K is the bulk modulus of compressibility.

   (a) Show that this system is hyperbolic and find the wave speeds.

   (b) Write a program to solve this system using Lax-Wendroff in original variables on $(0, 1)$ using a cell centered grid $x_j = (j - 1/2)h$ for $j = 1 \ldots N$. Write the code to use ghost cells, so that different boundary conditions can be changed by simply changing the values in the ghost cells.

   Set the ghost cells at the left by

$$\begin{aligned} p_0^n &= p_1^n \\ u_0^n &= -u_1^n, \end{aligned}$$

   and set the ghost cells on the right by

$$\begin{aligned} p_{N+1}^n &= \frac{1}{2}\left(p_N^n + u_N^n \sqrt{K\rho}\right) \\ u_{N+1}^n &= \frac{1}{2}\left(\frac{p_N^n}{\sqrt{K\rho}} + u_N^n\right). \end{aligned}$$

   Run simulations with different initial conditions. Explain what happens at the left and right boundaries.

   (c) Give a physical interpretation and a mathematical explanation of these boundary conditions.

2. A scheme is monotone preserving if the solution, u_j^n, is monotone in j for all n whenever the initial condition, u_j^0, is monotone in j. Show that if a scheme is TVD, then it is monotone preserving. Assume that the domain is the whole real line, that the solution satisfies the asymptotic boundary conditions $\lim_{j \to \pm\infty} u_j^n = U_{\pm\infty}$, and that the initial condition has bounded variation.

3. Write a program to solve the linear advection equation,

$$u_t + a u_x = 0,$$

   on the unit interval using a finite volume method of the form

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{h}\left(F_{j+1/2} - F_{j-1/2}\right).$$

Use the numerical flux function

$$F_{j-1/2} = F^{\text{up}}_{j-1/2} + \frac{|a|}{2}\left(1 - \left|\frac{a\Delta t}{h}\right|\right)\delta_{j-1/2},$$

where $F^{\text{up}}_{j-1/2}$ is the upwinding flux,

$$F^{\text{up}}_{j-1/2} = \begin{cases} a\,u_{j-1} & \text{if } a > 0 \\ a\,u_j & \text{if } a < 0, \end{cases}$$

and $\delta_{j-1/2}$ is the limited difference. Let $\Delta u_{j-1/2} = u_j - u_{j-1}$ denote the jump in u across the edge at $x_{j-1/2}$. The limited difference is

$$\delta_{j-1/2} = \phi\left(\theta_{j-1/2}\right)\Delta u_{j-1/2},$$

where

$$\theta_{j-1/2} = \frac{\Delta u_{J_{\text{up}}-1/2}}{\Delta u_{j-1/2}},$$

and

$$J_{\text{up}} = \begin{cases} j-1 & \text{if } a > 0 \\ j+1 & \text{if } a < 0 \end{cases}.$$

Note that you will need two ghost cells on each end of the domain. Write your program so that you may choose from the different limiter functions listed below.

Upwinding	$\phi(\theta) = 0$
Lax-Wendroff	$\phi(\theta) = 1$
Beam-Warming	$\phi(\theta) = \theta$
minmod	$\phi(\theta) = \text{minmod}(1, \theta)$
superbee	$\phi(\theta) = \max(0, \min(1, 2\theta), \min(2, \theta))$
MC	$\phi(\theta) = \max(0, \min((1+\theta)/2, 2, 2\theta))$
van Leer	$\phi(\theta) = \dfrac{\theta + \lvert\theta\rvert}{1 + \lvert\theta\rvert}$

The first three are linear methods that we have already studied, and the last four are high-resolution methods.

Solve the advection equation with $a = 1$ with periodic boundary conditions for the different initial conditions listed below until time $t = 5$ at Courant number 0.9.

(a) Wave packet: $u(x,0) = \cos(16\pi x)\exp(-50(x-0.5)^2)$.

(b) Smooth, low frequency: $u(x,0) = \sin(2\pi x)\sin(4\pi x)$.

(c) Step function: $u(x,0) \begin{cases} 1 & \text{if } |x - 1/2| < 1/4 \\ 0 & \text{otherwise} \end{cases}$.

Compare the results with the exact solution, and comment on the solutions generated by the different methods. How do the different high-resolution methods perform in the different tests? What high-resolution method would you choose to use in practice?