Lecture 14

Introduction

Continue from last time:
upwinding-idea: one-sided spatial differencing:

$u_t + au_x = 0$	
$a > 0$	characteristic to the right
$a < 0$	characteristic to the left

$$u_j^{n+1} = \begin{cases} u_j^{n+1} = u_j^n - \frac{a\Delta t}{h}(u_j^n - u_{j-1}^n) & \text{if } a > 0 \\ u_j^{n+1} = u_j^n - \frac{a\Delta t}{h}(u_{j+1}^n - u_j^n) & \text{if } a < 0 \end{cases}$$

Use the piecewise defined scheme for variable coefficient (a chenging sign)

Accuracy of upwind

LTE=$O(\Delta t) + O(h) \rightarrow$ first order space adn time.

Is it stable?

$$g(\xi) = 1 - \nu(1 - e^{-i\xi h}) = (1 - \nu) + \nu e^{-i\xi h}$$

Assume $a > 0$ $\nu \leq 1$ for stability for CFL, Upwinding is stable if $\nu = \frac{a\Delta t}{h} \leq 1$

getting second order in space and time

We chose BCD and interpolate to Q to get Lax-Wendroff method. If we choose ABC and interpolate to Q, We get Beam Warming.

More traditional derivation of these methods based on a Taylor expansion

$$u(x, t + \Delta t) = u(x, t) + \Delta t u_t(x, t) + \frac{\Delta t^2}{2} u_{tt}(x, t) + O(\Delta t^3)$$

Notice $u_t = -au_x$ Thus

$$u(x, t + \Delta t) = u(x, t) - a\Delta t u_x + O(\Delta t^2)$$

To get Higher accuracy, we need to include u_{tt} term, we use the PDE to replace u_{tt} with

$$u_{tt} = \frac{\partial}{\partial t}(-au_x) = -au_{xt} = -a(u_t)_x = -a(-au_x)_x = a^2 u_{xx}(a \cdot constant)$$

$$u(x, t + \Delta t) = u - a\Delta t u_x + \frac{\Delta t^3 a^2}{2} u_{xx} + O(\Delta t^3)$$

we will discretize the u_x and u_{xx} terms.
Lax-Wendroff use the centered diffence for both.

$$u_j^{j+1} = u_j^n - \frac{a\Delta t}{2h}(u_{j+1}^n - u_{j-1}^n) + \frac{a^2\Delta t}{2h^2}(u_{j+1}^n - 2u_j + u_{j-1}^n)$$

LTE is $O(\Delta t^2) + O(h^2)$

Beam Warming:
we use sided upwind difference for u_x and u_{xx}
if $a > 0$ then $u_j^{n+1} = u_j^n - \frac{a\Delta t}{2h}(3u_j^n - 4u_{j-1}^n + u_{j-2}^n) + \frac{a^2\Delta t^2}{2h^2}(u_j^n - 2u_{j-1}^n + u_{j+2}^n)$
LTE=$O(\Delta t^3) + O(\Delta th) + O(h^2)$

stability of LCU:
we need $|\nu| \leq 1$ from CFL

$$|g(\xi)|^2 = 1 - 4\nu^2(1-\nu^2)\sin^4(\frac{\xi h}{2})$$

$$\Rightarrow |g(\xi)|^2 = 1$$

$$g(\frac{\pi}{1}\frac{2}{h}) = 1 - 4\nu^2(1-\nu^2) = 1 - 4\nu^2 + 4\nu^4 = (1-2\nu^2)^2$$

$$\Rightarrow |1-2\nu^2| \leq 1$$

$$\Rightarrow -1 \leq 1 - 2\nu^2 \leq 1$$

$$\Rightarrow \nu \leq 1$$

$$\Rightarrow |\nu \leq 1|$$

for stability.