MAT 228B

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Lecture 3

Given the ODE

$$\begin{cases} y' = f(y) \\ y(0) = y_0 \end{cases}$$

Forward Euler discretization gives

$$\frac{y^{n+1} - y^n}{\Delta t} = f(y^n)$$

$$y^{n+1} = y^n + \Delta t f(y^n)$$

Method produce a sequence of values

$$y^n \approx y(\Delta t n)$$

Absolute Stability

Apply the method to

$$y' = \lambda y$$

where $\lambda \in C$

Let $z = \lambda \Delta t$. z is in the region of absolute stability if $y^n \to 0$ as $n \to \infty$. The region of absolute stability for forward Euler

$$y^{n+1} = y^n + \Delta t \lambda y^n$$

$$y^{n+1} = (1+z) y^n$$

$$y^n = (1+z)^n y_0$$

Absolutely stable when $|1+z| < 1$ i.e., z is in the disc of radius one centered at $z = -1$

Consider forward Euler for the head equation. $(D = 1)$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{D}{h^2}(u_{j-1}^n - 2u_j^n + u_{j+1}^n)$$

Method of line (MOL)

$$\frac{u^{n+1} - u^n}{\Delta t} = L u^n$$

We want to pick the time step so that I am in the region of abolute stability. We need

$$|1 + \lambda \Delta t| < 1$$

for all λ, eigenvalues of L

Eigenvalues of L are

$$\lambda_k = \frac{2}{h^2}(cos(k\pi h) - 1)$$

where $k = 1, ..., N$ and $h = \frac{1}{N+1}$

These eigenvalues are real and negative. The biggest eigenvalue (in modulus) is

$$\lambda_N = \frac{2}{h^2}(cos(N\pi h) - 1)$$

$$\lambda_n \approx \frac{-4}{h^2}$$

We need $|1 + \lambda \Delta t| < 1$, so we need

$$\Delta t(\frac{-4}{h^2}) > -2$$

$$\Delta t < \frac{h^2}{2}$$

Forward Euler is stable if this is satisfied.

If $D \neq 1$,

$$\Delta t < \frac{Dh^2}{2}$$

Why is forward Euler unstable for advection equation with centered difference in space?

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a}{2h}(u_{j+1}^n - u_{j-1}^n) = 0$$

Consider the spatially periodic domain. The centered difference operator D_0 has matrix form

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

The eigenvectors are of the form

$$u_j^{(k)} = e^{2\pi i k x_j}$$

$$\frac{u_{j+1}^{(k)} - u_{j-1}^{(k)}}{2h} = \frac{e^{2\pi i k x_{j+1}} - e^{2\pi i k x_{j-1}}}{2h} = (\frac{e^{2\pi i k h} - e^{-2\pi i k h}}{2h})e^{2\pi i k x_j} = \frac{isin(2\pi k h)}{h}e^{2\pi i k x_j}$$

Eigenvalue is

$$\frac{isin(2\pi k h)}{h}$$

Because it is pure imaginary, $\lambda \Delta t$ is never in the region of absolute stability.

The (discretized) diffusion equation is an example of a stiff equation. For this problem there is no need to resolve the fast time scales (high frequency modes).

Now consider the backward Euler,

$$\frac{y^{n+1} - y^n}{\Delta t} = f(y^{n+1})$$

This is our first example of implicit method. We need to solve for y^{n+1}.

Calculate the region of absolute stability

$$y' = \lambda y$$

$$\frac{y^{n+1} - y^n}{\Delta t} = \lambda y^{n+1}$$

$$y^{n+1} - \Delta t \lambda y^{n+1} = y^n$$

$$(1 - z) y^{n+1} = y^n$$

$$y^{n+1} = \frac{1}{1 - z} y^n$$

$$y^n = (\frac{1}{1 - z})^n y^0$$

Region of absolute stability is

$$|\frac{1}{1 - z}| < 1$$

$$|z - 1| > 1$$

which is the region outside the unit disc centered at 1. Backward Euler for heat equation is unconditionally stable. This is an example of an A-stable method (the whole left half plane is in the region of absolute stability).

Both forward Euler and backward Euler are first order accurate in time. The local truncation error is $O(\Delta t)$.

Let $u(x, t)$ be the solution to $u_t = D u_{xx}$. The local truncation error for forward Euler is

$$\tau = \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = D(\frac{u(x - h, t) - 2u(x, t) + u(x + h, t)}{h^2})$$

Let u = u(x,t)

$$\tau = \frac{u + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + \frac{\Delta t^3}{6} u_{ttt} + O(\Delta t^4) - u}{\Delta t} - D(u_{xx} - \frac{h^2}{12} u_{xxxx} + O(h^4))$$

$$\tau = u_t + \frac{\Delta t^2}{2} u_{tt} + O(\Delta t^2) - D u_{xx} - \frac{D h^2}{12} u_{xxxx} + O(h^4)$$

$$\tau = \frac{\Delta t^2}{2} u_{tt} - \frac{D h^2}{12} u_{xxxx} + O(\Delta t^2) + O(h^4)$$

which is first order in time and second order in space.