MAT 228B

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Lecture 3

We want to find a numerical solution to the given ODE

$$\begin{cases} y' = f(y) \\ y(0) = y_0 \end{cases}.$$

The simplest scheme is forward Euler:

$$\frac{y^{n+1} - y^n}{\Delta t} = f(y^n),$$

$$y^{n+1} = y^n + \Delta t f(y^n).$$

This method produces a sequence of values that approximate the solution to the ode at discrete time points:

$$y^n \approx y(\Delta t n).$$

Absolute Stability

We want to investigate the stability of time stepping schemes. To do this, we apply the method to the problem

$$y' = \lambda y$$

where $\lambda \in C$.

Let $z = \lambda \Delta t$. z is in the region of absolute stability if $y^n \to 0$ as $n \to \infty$. The region of absolute stability for forward Euler is calculated below.

$$y^{n+1} = y^n + \Delta t \lambda y^n$$

$$y^{n+1} = (1 + z) y^n$$

$$y^n = (1 + z)^n y_0$$

Thus the method is absolutely stable when $|1 + z| < 1$, i.e., z is in the disc of radius one centered at $z = -1$.

Consider forward Euler for the heat equation: $u_t = b u_{xx}$.

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{b}{h^2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right).$$

This can be written as

$$\frac{u^{n+1} - u^n}{\Delta t} = L u^n.$$

We want to pick the time step in the region of absolute stability for forward Euler. For this we need

$$|1 + \lambda \Delta t| < 1$$

for all λ, eigenvalues of L.

Eigenvalues of L are

$$\lambda_k = \frac{2b}{h^2}\big(\cos(k\pi h) - 1\big)$$

where $k = 1, ..., N$ and $h = \frac{1}{N+1}$.

These eigenvalues are real and negative. The biggest eigenvalue (in modulus) is

$$\lambda_N = \frac{2b}{h^2}\big(\cos(N\pi h) - 1\big),$$

$$\lambda_N \approx \frac{-4b}{h^2}.$$

The condition $|1 + \lambda \Delta t| < 1$, is satisfied whenever

$$\Delta t \left(\frac{-4b}{h^2}\right) > -2,$$

$$\Delta t < \frac{h^2}{2b}.$$

Therefore, forward Euler is stable if this bound on the time step holds.

Advection Equation

Question: Why is forward Euler unstable for advection equation with centered difference in space?

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a}{2h}(u_{j+1}^n - u_{j-1}^n) = 0$$

Consider the spatially periodic domain. The centered difference operator, D_0, is of the form

$$D_0 = \frac{1}{2h}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

The eigenvectors are of the form

$$u_j = e^{2\pi i k x_j}.$$

Substituting this expression into the centered difference,

$$\frac{u_{j+1} - u_{j-1}}{2h} = \frac{e^{2\pi i k x_{j+1}} - e^{2\pi i k x_{j-1}}}{2h} = \left(\frac{e^{2\pi i k h} - e^{-2\pi i k h}}{2h}\right) e^{2\pi i k x_j} = \frac{i \sin(2\pi k h)}{h} e^{2\pi i k x_j}.$$

Therefore the eigenvalues are

$$\frac{i \sin(2\pi k h)}{h}.$$

Because all eigenvalues are pure imaginary, $z = \lambda \Delta t$ is never in the region of absolute stability.

Backward Euler

The (discretized) diffusion equation is an example of a stiff equation. Stiff equations are characterized by multiple time scales in which the ratio of the slow time scale to the fast time scale is large. For this problem there is no need to resolve the fast time scales (high frequency modes).

Now consider backward Euler,

$$\frac{y^{n+1} - y^n}{\Delta t} = f(y^{n+1}).$$

This is a backward difference in time. This is our first example of implicit method, meaning that we need to solve an equation to find y^{n+1}.

Calculating the region of absolute stability

$$\begin{aligned}
y' &= \lambda y \\
\frac{y^{n+1} - y^n}{\Delta t} &= \lambda y^{n+1} \\
y^{n+1} - \Delta t \lambda y^{n+1} &= y^n \\
(1-z) y^{n+1} &= y^n \\
y^{n+1} &= \frac{1}{1-z} y^n \\
y^n &= \left(\frac{1}{1-z} \right)^n y^0
\end{aligned}$$

The region of absolute stability is

$$\begin{aligned}
\left| \frac{1}{1-z} \right| &< 1; \\
|z-1| &> 1.
\end{aligned}$$

This is the region outside the unit disc centered at 1. Backward Euler for the heat equation is unconditionally stable. This is an example of an A-stable method (the whole left half plane is in the region of absolute stability).

Time Accuracy

Both forward Euler and backward Euler are first-order accurate in time. The local truncation error is $O(\Delta t)$.

Let $u(x,t)$ be the solution to $u_t = b u_{xx}$. The local truncation error for forward Euler is

$$\tau = \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} - b \left(\frac{u(x-h,t) - 2u(x,t) + u(x+h,t)}{h^2} \right).$$

Let $u = u(x,t)$. Taylor expanding in both time and space gives,

$$\begin{aligned}
\tau &= \frac{u + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + \frac{\Delta t^3}{6} u_{ttt} + O(\Delta t^4) - u}{\Delta t} - b \left(u_{xx} - \frac{h^2}{12} u_{xxxx} + O(h^4) \right) \\
&= u_t + \frac{\Delta t}{2} u_{tt} + O(\Delta t^2) - b u_{xx} - \frac{bh^2}{12} u_{xxxx} + O(h^4) \\
&= (u_t - b u_{xx}) + \frac{\Delta t}{2} u_{tt} + O(\Delta t^2) - \frac{bh^2}{12} u_{xxxx} + O(h^4) \\
&= \frac{\Delta t}{2} u_{tt} - \frac{bh^2}{12} u_{xxxx} + O(\Delta t^2) + O(h^4),
\end{aligned}$$

which is first order in time and second order in space. Using a similar analysis, the local truncation error for backward Euler is

$$\tau = -\frac{\Delta t}{2} u_{tt} - \frac{bh^2}{12} u_{xxxx} + O(\Delta t^2) + O(h^4).$$