LECTURE NOTES 01/15/09

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There isn't any accuracy gain from using Backward Euler, just a gain in stability. What if we use a centered difference for the time derivative? We should be able to get second order accuracy. For simplicity consider $y' = f(y)$. So we get

$$\frac{y^{n+1} - y^{n-1}}{2\,\Delta t} = f(y^n)\ ,$$

the "midpoint method", which is second order accurate in time. This is a multilevel method (because we need the solution at two different times). This is not used much, since it has a very restricted stability region.

We can average F.E. and B.E. to get the trapezoidal method:

$$\frac{y^{n+1} - y^{n}}{\Delta t} = \frac{1}{2}\left(f(y^n) + f(y^{n+1})\right)\ ,$$

which is also second order accurate in time.Now we'll find the region of Absolute Stability for the trapezoidal rule:

$$\frac{y^{n+1} - y^{n}}{\Delta t} = \frac{\lambda}{2}\left(f(y^n) + f(y^{n+1})\right)\ .$$

If $z = \Delta t\ \lambda$, we must have

$$\frac{|\frac{z}{2} + 1|}{|\frac{z}{2} - 1|} < 1\ .$$

This is equivalent to $|z+2| \ < \ |z-2|$, hence the region of Absolute Stability is $Re(z) < 0$. We conclude that the trapezoidal region is A-stable. It is no harder to apply trapezoidal method than B.E.

Trapezoidal Rule applied to Heat/Diffusion equation is called "Crank-Nicolson". Other types of methods for ODE are "Runge-Kutta Methods" and "Multistep Methods".

Runge Kutta:

It's a 2-step, multistage method.Example of 2-stage explicit 2^{nd} order method:

$$\begin{aligned} y^* &= y^n + \Delta t\ f(y^n) \quad (Predictor) \\ y^* &= y^n + \Delta t\ \left(f(y^n) - f(y^*)\right) \end{aligned}$$

General r-stage method for $y' = f(t, y)$. The i-th stage

$$Y_i = y^n + \Delta t \sum_{j=1}^{r} A_{ij} f(t_n + c_j \Delta t_j y_j)$$

$$Y_i = y^n + \Delta t \sum_{j=1}^{n} b_j f(t_n + c_j \Delta t\ ,\ y_j)$$

A is the RK matrix, $\vec{b}$ are RK weights, $\vec{c}$ are the nodes. If A is lower triangular (strictly), the method is explicit. If A is lower triangular (non-zero on diagonal), the method is diagonally implicit.
Butcher table defines method:

$$\begin{array}{c} \vec{c}\ A \\ \vec{b}^T \end{array}$$

Linear Multistep Method:
r-step LMM has the form

$$\sum_{j=1}^{n} \alpha_j y^{n+j} = \Delta t \sum_{j=1}^{n} B_j f(y^{n+j})$$

looking r steps back in time.

Adam's Method:

$$\frac{y^{n+r} - y^{n+r-1}}{\Delta t} = \sum_{j=1}^{r} \beta_j f(y^{n+j})$$

If $\beta_r = 0$, the RHS does not involve y^{n+r}, so the method is explicit; Adams-Bashforth Method. If $\beta_r \neq 0$, it's implicit.

BDF-Method: $\beta_0 = \beta_1 = ... = \beta_{r-1} = 0$. 1-step BDF Method is Backward Euler. The 2-step BDF Method is

$$3y^{n+1} \ - 4y^n \ + y^{n-1} = 2\Delta t f(y^{n+1}) \ ,$$

where the LHS is a 3 point, 2^{nd} order accurate approximation to y'. It is also A-stable.

Talking more formally about Stability

Definition (Consistency, Stability and Convergence)
A numerical method is convergent, if for any fixed point $(x^*,\ t^*)$ in the domain $\Omega \times [0,1]$ we have $||u_j^n - u(x^*, t^*)|| \to 0$ whenever $x_j \to x^*$ and $t_n \to t^*$. $(h \to 0\ , \Delta t \to 0)$

There may be a restriction on how h and $\Delta t\ \to 0$. For example F.E. for the Heat equation as $h + \Delta t\ \to 0$ provided $\Delta t < \frac{h^2}{2\delta}$.

Sometimes take $\Delta t(h)$, by fixing ν, and letting h and Δt approach 0 by $\frac{\Delta t\, \delta}{h^2} = \nu$.

A numerical scheme is consistent if the local truncation error τ approaches 0 as Δt and h do so. If $\tau = O(\Delta t^p)\ +\ O(h^q)$, we say the method is p-th order in time and q-th order in space.

Lax Equivalence Theorem
For a linear method an linear PDE, stability and consistency implies convergence.