LECTURE NOTES 01/15/09

SHAD PEARSON AND MORITZ REINTJES

There isn't any accuracy gain from using Backward Euler, just a gain in stability. What if we use a centered difference for the time derivative? We should be able to get second order accuracy. For simplicity consider $y' = f(y)$. So we get

$$\frac{y^{n+1} - y^{n-1}}{2\,\Delta t} = f(y^n)\ ,$$

the "midpoint method" or "leapfrog method", which is second order accurate in time. This is a multilevel method (because it involves more that two time levels). This is not commonly used because it has a very restricted stability region.

Averaging forward Euler and backward Euler gives the trapezoidal method:

$$\frac{y^{n+1} - y^{n}}{\Delta t} = \frac{1}{2}\left(f(y^n) + f(y^{n+1})\right)\ ,$$

which is also second-order accurate in time. One can see that it is second-order in time examining the truncation errors of forward and backward Euler. Also this can be interpreted as a centered difference in time about the half time level $n + 1/2$.

To find the region of absolute stability for trapezoidal rule, examine the solution to

$$\frac{y^{n+1} - y^{n}}{\Delta t} = \frac{\lambda}{2}\left(y^n + y^{n+1}\right)\ .$$

If $z = \Delta t\ \lambda$, we must have

$$\frac{|\frac{z}{2} + 1|}{|\frac{z}{2} - 1|} < 1\ .$$

This is equivalent to $|z+2| \ < \ |z-2|$, which means that the region of absolute stability consists of all the points in the complex plane that are closer to -2 than to 2. Hence the region of Absolute Stability is $Re(z) < 0$. We conclude that the trapezoidal region is A-stable. It is no harder to apply trapezoidal method than B.E., and it is more accurate in time.

Trapezoidal Rule applied to Heat/Diffusion equation is called "Crank-Nicolson". It is widely used because it is second-order in space and time and unconditionally stable.

Other types of methods for ODE are "Runge-Kutta Methods" and "Multistep Methods". These are considered next.

Runge Kutta:

Runge Kutta methods are one-step/multistage methods. An example of a 2-stage explicit 2^{nd} order method:

$$y^* \ = \ y^n + \Delta t\ f(y^n)$$

$$y^* \quad = \quad y^n + \frac{\Delta t}{2}\ \left(f(y^n) - f(y^*)\right).$$

General r-stage method for $y' = f(t, y)$:

$$Y_i = y^n + \Delta t \sum_{j=1}^{r} A_{ij} f(t_n + c_j \Delta t_j,\ Y_j), \quad i = 1 \ldots r$$

$$y^{n+1} = y^n + \Delta t \sum_{j=1}^{r} b_j f(t_n + c_j \Delta t\ ,\ Y_j).$$

A is the RK matrix, $\vec{b}$ are the RK weights, and $\vec{c}$ are the RK nodes. If A is lower triangular (strictly), the method is explicit. If A is lower triangular (non-zero on diagonal), the method is diagonally implicit.

Butcher table defines the method method: $\begin{array}{c|c} \vec{c} & \text{A} \\ \hline & \vec{b}^T \end{array}$

An example is classical RK4: $\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 0 & 1/2 & & \\ 1/2 & 0 & 0 & 1/2 & \\ 1 & 0 & 0 & 0 & 1 \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$

Linear Multistep Method:
An r-step LMM has the form

$$\sum_{j=1}^{r} \alpha_j y^{n+j} = \Delta t \sum_{j=1}^{r} \beta_j f(y^{n+j})$$

This uses the solution at the past r levels of time to update the solution.

Adams Methods:

$$\frac{y^{n+r} - y^{n+r-1}}{\Delta t} = \sum_{j=1}^{r} \beta_j f(y^{n+j})$$

If $\beta_r = 0$, the RHS does not involve y^{n+r}, so the method is explicit. Explicit Adams methods are called Adams-Bashforth Methods. If $\beta_r \neq 0$, the method is implicit. Implicit Adams methods are called Adams-Moulton methods. The one-step AB method is forward Euler. The one-step AM method is trapezoidal rule.

BDF-Methods:

$$\sum_{j=1}^{r} \alpha_j y^{n+j} = \Delta t \beta_r f(y^{n+r})$$

These are multistep methods for which $\beta_0 = \beta_1 = ... = \beta_{r-1} = 0$. 1-step BDF Method is Backward Euler. The 2-step BDF Method is

$$3y^{n+1}\ - 4y^n\ + y^{n-1} = 2\Delta t f(y^{n+1})\ ,$$

where the LHS is a 3 point, one-sided, 2^{nd} order accurate approximation to y'. It is also A-stable.

Consistency, Stability and Convergence

Definition

A numerical method is convergent, if for any fixed point $(x^*,\ t^*)$ in the domain $\Omega \times [0,1]$ we have $||u_j^n - u(x^*, t^*)|| \to 0$ whenever $x_j \to x^*$ and $t_n \to t^*$. $(h \to 0\ , \Delta t \to 0)$

There may be a restriction on how h and $\Delta t\ \to 0$. For example F.E. for the heat equation converges as $h, \Delta t\ \to 0$ provided $\Delta t < h^2/(2b)$. Sometimes we write $\Delta t(h)$ to indicate that the time step is some function of the space step. For example, we may let h go to zero and fix the ratio $\nu = b\Delta t/h^2 < 1/2$.

Definition

A numerical scheme is consistent if the local truncation error, τ, approaches 0 as $h, \Delta t\ \to 0$. If $\tau = O(\Delta t^p)\ +\ O(h^q)$, we say the method is p-th order in time and q-th order in space.

Lax Equivalence Theorem

For a linear difference method approximation to a linear, well-posed PDE, stability and consistency imply convergence.