**Math 228C**
**Homework 1**
**Due Thursday, 04/23/09.**

1. Show that for smooth initial data the scalar conservation law

$$q_t + (f(q))_x = 0$$

develops a shock at time

$$T = \frac{-1}{\min\big(f''(q(x,0))q_x(x,0)\big)}$$

if this quantity is positive.

2. (a) Determine the analytic solution to the inviscid Burgers equation

$$u_t + \frac{1}{2}\left(u^2\right)_x = 0,$$

with initial data

$$u(x,0) = \begin{cases} 2 & \text{if } |x| \le 1 \\ -1 & \text{otherwise} \end{cases}$$

on the real line.

(b) Write a program to solve this problem using Godunov's method up to time $t = 2$, and show that your numerical solution captures the correct shock speed.

(c) Write a program to solve this problem using a high-resolution method. Describe your method and compare the numerical solution with the exact solution and with the numerical solution from Godunov's method.

3. In this problem you will solve the viscous Burgers equation

$$u_t + uu_x = \mu u_{xx}$$

on $(0,1)$ with boundary conditions $u(0) = 1.1$ and $u(1) = -1$ and initial data $u(x,0) = 1.05\cos(\pi x) + 0.05$ up to time $t = 5$. The solution will develop into a traveling wave. The steepness of the wave depends on the size of the viscosity.

(a) Discretize the $uu_x$ terms explicitly (either forward Euler or 2-step Adams-Bashforth) in time and use a centered difference in space. Discretize the viscous terms using Crank-Nicolson in time. This method will work well provided $\mu$ is large enough. Experiment with different values of $\mu$ to determine when this method is useful (this will depend on the grid spacing that you choose).

(b) Use methods for hyperbolic conservation laws to discretize the nonlinear terms rather than the centered difference. Describe your method and demonstrate that it is effective for small values of $\mu$.