Math 228C
Homework 1
Due Wednesday, 4/17

1. Consider the advection equation
$$u_t + au_x = 0$$
on the interval $[0, 1)$ with periodic boundary conditions. Space in discretized as $x_j = j\Delta x$ for $j = 0 \dots N-1$, so that $\Delta x = 1/N$. Discretize the spatial derivative with the second-order centered difference operator.

   (a) For simplicity, assume N is odd. The eigenvectors of the centered difference operator are
   $$v_j^k = \exp\left(2\pi i k x_j\right),$$
   for $k = -(N-1)/2 \dots (N-1)/2$. Compute the eigenvalues.

   (b) Derive a time step restriction on a method-of-lines approach which uses classical fourth-order Runge-Kutta for time stepping.

2. Consider the following PDE.
$$u_t = 0.01\, u_{xx} + 1 - \exp(-t), \quad 0 < x < 1$$
$$u(0,t) = 0 \quad u(1,t) = 0$$
$$u(x,0) = 0$$

   (a) Write a program to solve the problem using Crank-Nicolson up to time $t = 1$, and perform a refinement study that demonstrates that the method is second-order accurate in space and time.

   (b) Write a program to solve the problem using BDF2 up to time $t = 1$, and perform a refinement study that demonstrates that the method is second-order accurate in space and time.

3.
$$u_t = u_{xx}, \quad 0 < x < 1$$
$$u(0,t) = 1, \quad u(1,t) = 0$$
$$u(x,0) = \begin{cases} 1 & \text{if } x < 0.5 \\ 0 & \text{if } x \geq 0.5 \end{cases}$$

   (a) Use Crank-Nicolson with grid spacing $\Delta x = 0.02$ and time step 0.1 to solve the problem up to time $t = 1$. Comment on your results. What is wrong with this solution?

   (b) Experiment with smaller time steps. How small does the time step need to be to get reasonable results?

   (c) Give a mathematical argument to explain the nonphysical behavior you observed in the numerical solution.

   (d) Repeat the calculation using BDF2 in place of Crank-Nicolson. Discuss what is different about the results, and explain mathematically why the two schemes perform so differently on this problem.