Math 228C
Homework 2
Due Thursday, 06/11/09.

1. (a) Let D and G be the discrete divergence and gradient operators arising from a staggered (MAC) discretization of a rectangular domain. Show that $D = -G^{\mathrm{T}}$ in the sense that for all scalar fields ψ and discrete vector fields $\boldsymbol{v}$ that have zero normal component on the boundary

$$\langle \boldsymbol{v}, G\psi \rangle_v = \langle -D\boldsymbol{v}, \psi \rangle_s,$$

where these are the standard inner products on discrete vectors (v) and scalars (s), respectively. Use this adjoint property to show that discrete gradient fields are orthogonal to discretely divergence-free fields that have zero normal component on the boundary.

(b) Consider the following projection method to advance the velocity and pressure with homogeneous Dirichlet conditions for the velocity.

$$\frac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} + Gp^n = \frac{1}{2}\left(L\boldsymbol{u}^n + L\boldsymbol{u}^*\right)$$
$$\Delta t DG\phi = Du^*$$
$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^* - \Delta t G\phi$$
$$p^{n+1} = p^n + 2\phi,$$

where D and G are the discrete divergence and gradient operators, respectively. Assume that D and G are related by $D = -G^{\mathrm{T}}$ and that L is negative definite. Show that

$$\left|\left|\boldsymbol{u}^{n+1}\right|\right|_2^2 + \frac{\Delta t^2}{4}\left|\left|Gp^{n+1}\right|\right|_2^2 \leq \left|\left|\boldsymbol{u}^n\right|\right|_2^2 + \frac{\Delta t^2}{4}\left|\left|Gp^n\right|\right|_2^2.$$

Hint: take the dot product of the first equation with $\boldsymbol{u}^* + \boldsymbol{u}^n$.

(c) Show that velocity and pressure gradient generated by the projection method above (including forcing and nonzero boundary velocities) converges to the solution of

$$\boldsymbol{u}_t + \nabla p = \Delta \boldsymbol{u} + \boldsymbol{f}$$
$$\nabla \cdot \boldsymbol{u} = 0.$$

2. (a) Use the level set method to simulate the movement of a circle expanding in the normal direction with constant speed. Run the simulation to a fixed point in time and show that your code converges as the grid is refined. Compare the error at the end of the simulation for different initializations of the level set function (e.g. signed distance function, characteristic function, smoothed characteristic function). You can use the MATLAB functions `contour` or `contourc` to locate the zero level set.

(b) Consider the problem of locating an advancing front moving in the normal direction at a constant speed. Suppose that the initial front consists of two circles of radius 0.5, with centers as (-1,0) and (1,0). Use your level set code to solve for the location of the front for some time beyond the point at which the two fronts merge. Show that your method captures the correct solution. Note that you can construct the analytic solution at time t using the level sets of the signed distance function at time zero. The signed distance function for the two circle problem is just the minimum of the signed distance function for each circle.