Solution key, Midterm II.

1) a) $r^2+2r+1 \Rightarrow (r+1)^2 \Rightarrow c_1 e^{-t} + c_2 t e^{-t}$

b) $r^2+r-6 \Rightarrow (r+3)(r-2) \Rightarrow c_1 e^{-3t} + c_2 e^{2t}$

c) $r^2+2r+4 \Rightarrow \begin{array}{l} r_1 = -1+i\sqrt{3} \\ r_2 = -1-i\sqrt{3} \end{array} \Rightarrow c_1 e^{-t}\cos(\sqrt{3}t) + c_2 e^{-t}\sin(\sqrt{3}t)$

2) a) $\underbrace{x^2(x'')}_{0} - x(x+2) + (x+2)x = 0.$ ✓

$x^2(e^x+e^x+xe^x) - x(x+2)(e^x+xe^x) + (x+2)xe^x = 0.$ ✓

b) a fundamental set of solutions spans the entire solution space. Yes, they are linearly independent and hence $W \neq 0$, but this is only enough for the solution to be fundamental in 2-dim, i.e. -2nd order or a 2 eq. 1st order system.

c) Yes. $W[y_1, y_2] = \det \begin{bmatrix} x & xe^x \\ 1 & e^x+xe^x \end{bmatrix} = x^2e^x \neq 0 \quad (x>0)$.

3) a) Looking for particular solution - not general.

Try $y(t) = (At+B)e^t$ (Ae^t doesn't work),→

$y'(t) = (At+A+B)e^t$

$y''(t) = (At+2A+B)e^t$

plug in $\Rightarrow (3At+3A+3B)e^t = 3te^t$

compare coefficients, $3A+3B=0$

$3At = 3te^t \Rightarrow A=1$

$B=-1$

$\Rightarrow Y(t) = (t-1)e^t$

b) - example 2, p.177.

4) a) $te^t - (1+t)e^t + e^t = 0.$ ✓

b) Let $y(t) = e^t \cdot v(t)$.

$\Rightarrow y'(t) = e^tv' + e^tv, \quad y''(t) = e^tv'' + 2e^tv' + e^tv$

plug in $\Rightarrow$ $ty'' - (t+1)y' + y$

$t\,(e^t v'' + 2e^t v' + e^t v) - (1+t)(e^t v' + e^t v) + e^t v = 0$

$= \underbrace{(tv'' + (t-1)v')}_{0}\, e^t = 0$

$\Rightarrow \quad 0 = 0$

$\Rightarrow tv'' + (t-1)v' = 0$

$\Rightarrow v'' + \frac{t-1}{t}v' = 0$ — in teg. factor method.

$\Rightarrow \mu(t) = \frac{e^t}{t}$

reverse power rule $\Rightarrow$ $\left(\frac{e^t}{t} v'\right) = 0$ — this means $\frac{e^t}{t} \cdot v'$ is a constant.

$\Rightarrow v' = c_1 \frac{t}{e^t}$.

$\Rightarrow v = C_1 (t+1) e^{-t} + C_2$

$\Rightarrow y(t) = e^t c_1 (t+1) \cdot e^{-t} + C_2 e^t$

$\Rightarrow \boxed{t+1}$

5) Let ~~$L[y]$~~ $L_{p,q}[y] = y'' + p(t)y' + q(t)y$.

use linearity $\Rightarrow$ $L[c_1 y_1 + c_2 y_2 + 3y_3] = c_1 L[y_1] + c_2 L[y_2] + 3L[y_3]$

$= 3\left(\frac{1-t^2}{1+t^2}\right)$

use initial conditions, $y_1(0) = 0$ $\quad y_1'(0) = 1$

$y_2(0) = 1$ $\quad y_2'(0) = 0$

$y_3(0) = 1$ $\quad y_3'(0) = 0$

$\Rightarrow c_2 + 3 = 0$

$c_1 = 1$

$\Rightarrow t - 3e^{-t^2/2} + 3$

6) a) - look it up, page 139

b) $\pi^{2i} = e^{\ln \pi^{2i}} = e^{2i \ln \pi} = \cos(2\ln\pi) + i\sin(2\ln\pi)$