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Angelo Carino
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Chapter 3 Study Guide

-A 2nd order differential equation (DE) often has the form:

$$P(t)y'' + Q(t)y' + R(t)y = G(t) \qquad (1)$$

-Or dividing by P(t):

$$y'' + p(t)y' + q(t)y = g(t) \qquad (2)$$

-This is also known as the **differential operator** denoted by

$$L[y] = y'' + p(t)y' + q(t)y = g(t)$$

-If the 2nd order DE has the form of either equation (1) or (2), it is known as a **linear DE** otherwise it is a **non-linear DE**
-If g(t) = G(t) = 0, then the DE is considered to be **homogenous**, otherwise the DE is **nonhomogeneous**
-An initial value problem (IVP) consists of a DE, such as equation (1) or (2), together with a pair of initial conditions: $y(t_0) = y_0$ and $y'(t_0) = y'_0$

-When solving a homogeneous DE we usually consider that P(t), Q(t), and R(t) in equation (1) are constant, thus we obtain

$$ay'' + by' + cy = 0 \qquad (3)$$

-The **characteristic equation** using equation (3) is defined to be

$$ar^2 + br + c = 0$$

-Before we can actually solve a 2nd order DE, we would like to know if a solution exists, this brings us to the **Existence and Uniqueness Theorem:**

-If we have an IVP **with** the form of equation (2), and if p(t), q(t), and g(t) are continuous on an open interval I that contains t_0, then there exists a solution to the DE and it exists throughout the interval I

Example: Find largest interval in which the solution of the IVP exists

$$(t^2 - 3t)y'' + ty' - (t + 3)y = 0 \qquad y(1) = 2,\ y'(1) = 1$$

Step 1: Get IVP into form of equation (2)

$$y'' + (\,t / (t^2 - 3t)\,)y' - (\,t + 3 / t^2 - 3t)y = 0$$

Step 2: Find where p(t), q(t) and g(t) are continuous (or discontinuous)

$p(t) = 1 / (t - 3)$, $q(t) = -(t + 3) / t(t - 3)$, $g(t) = 0$

-Only points of discontinuity are at $t = 0$ and $t = 3$.

Step 3: Find largest interval I that contains t_0 ($t_0 = 1$ in this case)

We find that this largest interval is $0 < t < 3$

-Thus this interval guarantees that the solution exists and is in fact unique

-**Principle of Superposition:** if y_1 and y_2 are two solutions of the differential operator $L[y] = y'' + p(t)y' + q(t)y = 0$, then any linear combination $c_1y_1 + c_2y_2$ is also a solution for any c_1 and c_2 in **R** (any real number)

-We define the Wronskian of y_1 and y_2 as $W = y_1y'_2 - y'_1y_2$ (think of determinant)

Find the Wronskian for the given pair of functions
Let $y_1 = e^{2t}$, and $y_2 = e^{-3t}$

-Find derivatives, and plug into Wronskian equation
$y'_1 = 2e^{2t}$ and $y'_2 = -3e^{-3t}$
Therefore, $W[e^{2t}, e^{-3t}] =$ ~~(7/2)e^{t/2}~~ $-5e^{-t}$

-If, given a pair of initial conditions along with $L[y] = 0$, and that the Wronskian at t_0 is non-zero, then there exists a choice of c_1 and c_2 for which $y = c_1y_1 + c_2y_2$ satisfies the DE and the initial conditions.

-If there is a point t_0 where the Wronskian of y_1 and y_2 is nonzero then the family of solutions $y = c_1y_1 + c_2y_2$ with arbitrary coefficients c_1 and c_2 includes every solution of $L[y] = y'' + p(t)y' + q(t)y = 0$.

-$y = c_1y_1 + c_2y_2$ is known as the **general solution**, this along with a nonzero Wronskian at any t is known as the **fundamental set of solutions**

-If we have a general solution that satisfies $L[y] = 0$ and a nonzero Wronskian at any t, this also implies **linear independence**. If the Wronskian for all t is zero this implies **linear dependence.**

Determine whether the pair of functions are linearly independent
$f(t) = Cos(t)$, $g(t) = Sin(t)$

-If we can find that the Wronskian is nonzero at *any* t will imply that these two functions are linearly independent

$W(Cos(t), Sin(t)) = Cos^2(t) + Sin^2(t) = 1$, for all t.
Thus, these two functions linearly independent

-**Abel's Theorem** states that if y_1 and y_2 are solutions of the DE $L[y] = 0$ then the Wronskian of y_1 and y_2 is given by:

$W[y_1, y_2](t) = C\exp(-\int p(t)dt)$

-Where C is a constant that depends on y_1 and y_2

-When solving a 2nd order DE, it can either be homogeneous, **Case (1)**, or non-homogeneous, **Case (2)**

Case (1) – Homogeneous DE

-When solving a homogeneous DE we first find the characteristic equation

-Then we find the roots to that equation (via quadratic formula or factoring)

-There can be three possible cases for the roots:

Case A: Non-complex roots

Case B: Complex roots

Case C: Repeated roots

-Using the roots we form the general solution to the homogeneous DE

-If we have an IVP problem, find the general solution and the derivative of that solution. Plug in the initial conditions and solve for c_1 and c_2

Case A: Non-Complex Roots

-The general solution has the form: $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

-Where r_1 and r_2 are the roots to the characteristic equation

Example: Find general solution of the given DE

$y'' + 2y' - 3y = 0$

Step 1: Find characteristic equation

$r^2 + 2r - 3 = 0$

Step 2: Find the roots

$(r + 3)(r - 1) = 0; r = 1, -3$

Step 3: Form the general solution

$y = c_1 e^{t} + c_2 e^{-3t}$

Case B: Complex Roots

-The general solution has the form: $y = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t)$

-Where μ and λ are given by the roots of the characteristic equation

$r = \lambda \pm \mu i$ (solve for roots via quadratic formula)

Example: Find solution to DE

$y'' + 4y = 0; \quad y(0) = 0, y'(0) = 1$

Step 1: Find characteristic equation

$r^2 + 4 = 0$

Step 2: Find roots (use quadratic)

$\pm(-4*1*4)^{1/2}/2 = \pm 2i$

-Here we have **$\lambda = 0$ and $\mu = 2$**

Step 3: Form general solution

$y(t) = c_1 \text{Cos}(2t) + c_2 \text{Sin}(2t)$

Step 4: Find c_1 and c_2

-Find derivative. Plug in initial conditions, solve for c_1 and c_2

-We find that $c_1 = 0$ and $c_2 = ½$

-Thus the solution to this DE is: $y(t) = ½ \text{Sin}(2t)$

Case C: Repeated Roots

-The general solution has the form: $y = c_1 e^{rt} + c_2 v(t) e^{rt}$

-Where r is root of characteristic equation and $v(t) = t$

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Example: Find solution to DE

$$y'' + 4y' + 4y = 0$$

Step 1: Find characteristic equation

$$r^2 + 4r + 4 = 0$$

Step 2: Find roots

$$(r + 2)^2 = 0;\ r = -2$$

Step 3: Form general solution

$$y = c_1e^{-2t} + c_2\,te^{-2t}$$

Case (2) – Nonhomogeneous DE

-There are three methods to solving nonhomogeneous DE's:

(1) Reduction of Order
(2) Method of Undetermined Coefficients
(3) Variation of Parameters

-If Y_1 and Y_2 are two solutions of a nonhomogeneous equation then their difference is a solution of the corresponding homogeneous equation.

-The general solution of a nonhomogeneous equation can be written as

$$y = c_1y_1 + c_2y_2 + Y(t)$$

-Where $Y(t)$ is known as the **particular solution** and $c_1y_1 + c_2y_2$ is known as the **complementary solution,** which is the solution to the corresponding homogeneous DE

(1) Reduction of Order

-This method is only viable when we are already given a solution to the DE

-Given the first solution to the DE, y_1, to find a second solution let $y_2 = v(t)\,y_1(t)$

-Plug y_2 into DE

-Solve for $v(t)$

Example: Use reduction of order to find a second solution of given DE

$$t^2y'' - 4ty' + 6y = 0 \qquad y_1(t) = t^2$$

Step 1: Multiply $y_1(t)$ by $v(t)$ to obtain y_2

$$y_2 = v(t)\,t^2$$

Step 2: Plug y_2 into DE

$$t^2(t^2v'' + 4tv' + 2v) - 4t(t^2v' + 2tv) + 6t^2v = 0$$

-Collect terms and we get:

$$t^4v'' = 0$$

Step 3: Solve for $v(t)$

-We know that v'' must be 0 for that equation to hold.

-Thus $v'(t) = C$ and thus $v(t) = Ct + D$ (put all coefficients into one)

-Therefore $y_2 = Ct^3$

Note: In solving for $v(t)$ we will sometimes have to use separation of variables or the integrating factor method. To do this, for instance, make a substitution such as $y = v'$, and thus $y' = v''$.

(2) Method of Undetermined Coefficients

-This method involves making a guess about the form of the particular solution $Y(t)$, based on $g(t)$, but with the coefficients not specified. We then substitute

Y(t) into the DE and attempt to determine the coefficients to satisfy the DE. If we are successful then Y(t) is a solution to the DE, if we are not we then modify Y(t) and try again until we are successful.

-If g(t) is a product we must take into account all possible derivatives

-For sums of exponential, polynomials, etc, solve individually for each term then add all of them up.

-If g(t) is a solution of homogeneous DE then $Y(t) = tg(t)$ or $Y(t) = t^2g(t)$

Example: Find solution to DE

$y'' - 4y = 2e^{3t}$

Step 1: Guess Y(t) based on g(t)

$Y(t) = Ae^{3t}$

Step 2: Plug into DE and solve for coefficients

$9Ae^{3t} - 4Ae^{3t} = 2e^{3t}$

$9A - 4A = 2$

$A = 2/5$

Thus, $Y(t) = 2/5\ e^{3t}$

-If unsuccessful, modify Y(t) and try the method once more.

(3) Variation of Parameters

-More general way to find nonhomogeneous DE

-DE ***must*** be in form of equation (2)

-Find the complementary equation to the DE

-Find the Wronskian using the y_1 and y_2 from the complementary equation

-The particular solution is given by:

$$Y_P(t) = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)} dt$$

-The general solution is of the form: $y = c_1y_1 + c_2y_2 + Y(t)$

Example: Find the general solution to the DE

$y' - 2y' + y = e^t / (t^2 + 1)$

Step 1: Find the complementary solution

-Note: DE is already in form of equation (2)

-We find that the complementary solution is:

$y = c_1e^t + c_1te^t$

so we have $y_1 = e^t$ and $y_2 = te^t$

Step 2: Find the Wronskian

$W[e^t, te^t] = e^{2t}$

Step 3: Find the particular solution

-Plugging into the equation for Y(t) and simplifying we get:

$Y(t) = (^-1/2)\, e^t \ln(1 + t^2) + t\, e^t \tan^{-1}(t)$

Step 4: Form the general solution

$y(t) = c_1e^t + c_1te^t - (1/2)\, e^t \ln(1 + t^2) + t\, e^t \tan^{-1}(t)$