1.(5 points each) Say if the following ODE are **linear** or **non-linear**. By use of the theorems given in class, what can you say about the existence and uniqueness of the solutions? i.e. - when do they exist?

a) $(t+1)y' + \cos(t)\, y = e^t, \quad y(1) = -2.$

Linear $(y' + p(t)y = g(t))$ - we don't care about p, g being non-linear.

$$p(t) = \frac{\cos(t)}{t+1} \qquad g(t) = \frac{e^t}{t+1}$$

$t+1 \neq 0 \Rightarrow t \neq -1$

$-1 < t < \infty.$

b) $(y+1)y' + \cos(t)\, y = e^t, \quad y(1) = -2.$

non linear $(y \cdot y' \text{ term}).$

hence $y \neq -1$. but we don't know what t causes $y(t) = -1$, so a unique solution exists in some interval around $t_0 = 1$ $(y(1) = -2 \neq -1)$.

2. (8 points each) Find the general solutions of the following:

a) $y' = (1 - 2x)y^2$

separate $\frac{dy}{y^2} = (1-2x)dx$, integrate.-

$$\int \frac{dy}{y^2} = \int (1-2x)\,dx$$

$$= \frac{-1}{y} = x - x^2 + C$$

$$\Rightarrow \quad y = \frac{1}{x^2 - x + C}$$

b) $(x^2 + 1)y' + 3xy = 6x$

rewrite $y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$

already > 0 $\forall x$.

integ. factor $\Rightarrow e^{\int \frac{3x}{x^2+1}dx} = e^{\frac{3}{2}(\ln(x^2+1))} = (x^2+1)^{3/2}$

$$\Rightarrow \quad (x^2+1)^{3/2}y' + \left(\frac{3x}{x^2+1}\right)(x^2+1)^{3/2}y = \frac{6x}{\cancel{(x^2+1)}}(x^2+1)^{3/2}$$

$$= \int \left((x^2+1)^{3/2}y\right)' = \int 6x(x^2+1)^{1/2}dx$$

$$= (x^2+1)^{3/2}y = 2(x^2+1)^{3/2} + C \quad \Rightarrow \quad y = 2(x^2+1)^{1/2} + \frac{C}{(x^2+1)^{3/2}}$$

3. a) (10 points) Find the solution of the initial value problem (IVP)

$$ty' + 3y = \frac{1}{t}, \quad t > 0$$
$$y(1) = y_0.$$

$y' + \frac{3}{t}y = \frac{1}{t^2}$ $\Rightarrow$ Integrating factor

$\mu(t) = e^{\int \frac{3}{t} dt} = e^{3\ln t} = (e^{\ln t})^3 = t^3.$

$\Rightarrow t^3 y' + 3t^2 y = t$

$\int (t^3 y)' = \int t\, dt \Rightarrow t^3 y = \frac{t^2}{2} + c$

$\Rightarrow y = \frac{1}{2t} + \frac{c}{t^3}$

use $y(1) = y_0$,

$y_0 = \frac{1}{2} + c \Rightarrow c = y_0 - \frac{1}{2}$

$y = \frac{1}{2t} + \frac{y_0 - \frac{1}{2}}{t^3}$

b) (4 points) For what possible initial values of y_0 is the solution $y(t)$ equal to 0 for $t > 0$?

$y = \frac{1}{t}\left(\frac{1}{2} + \frac{y_0 - \frac{1}{2}}{t^2}\right) = 0$ if $\Rightarrow = 0$

$\Rightarrow \frac{y_0 - \frac{1}{2}}{t^2} < 0$

$\Rightarrow \boxed{y_0 < \frac{1}{2}}$

4. a) (10 points) Solve the initial value problem (IVP)

$$ty' + y^2 = 0,$$
$$y(1) = 1.$$

Separate, $\quad -\int \frac{dy}{y^2} = \int \frac{dt}{t}$

$$-\left(\frac{-1}{y}\right) = \ln|t| + C$$

$$\Rightarrow y = \frac{1}{\ln(t) + C}$$

$(t_0 > 0 \Rightarrow$ no need for absolute value$)$

use $y(1) = 1$, $\quad 1 = y = \frac{1}{\ln(1) + C} = \frac{1}{0 + C}$

$$C = 1$$

$$\Rightarrow \quad y = \frac{1}{\ln(t) + 1}$$

b) (4 points) What is the largest t-interval in which the solution exists?

$\ln(t) + 1 \neq 0 \Rightarrow \ln(t) = -1$ when $t = \frac{1}{e}$

$$\Rightarrow \frac{1}{e} < t < \infty$$

5. a) (10 points) Draw a direction field for the following differential equation,

$$y' - y = 1 + 3\sin(t), \quad y(0) = y_0.$$

b) (10 points) For what value of y_0 does the solution remain finite as $t \to \infty$?

need to solve IVP.

$y' - y = 1 + 3\sin(t)$

$\mu(t) = e^{-\int 1\,dt} = e^{-t}$

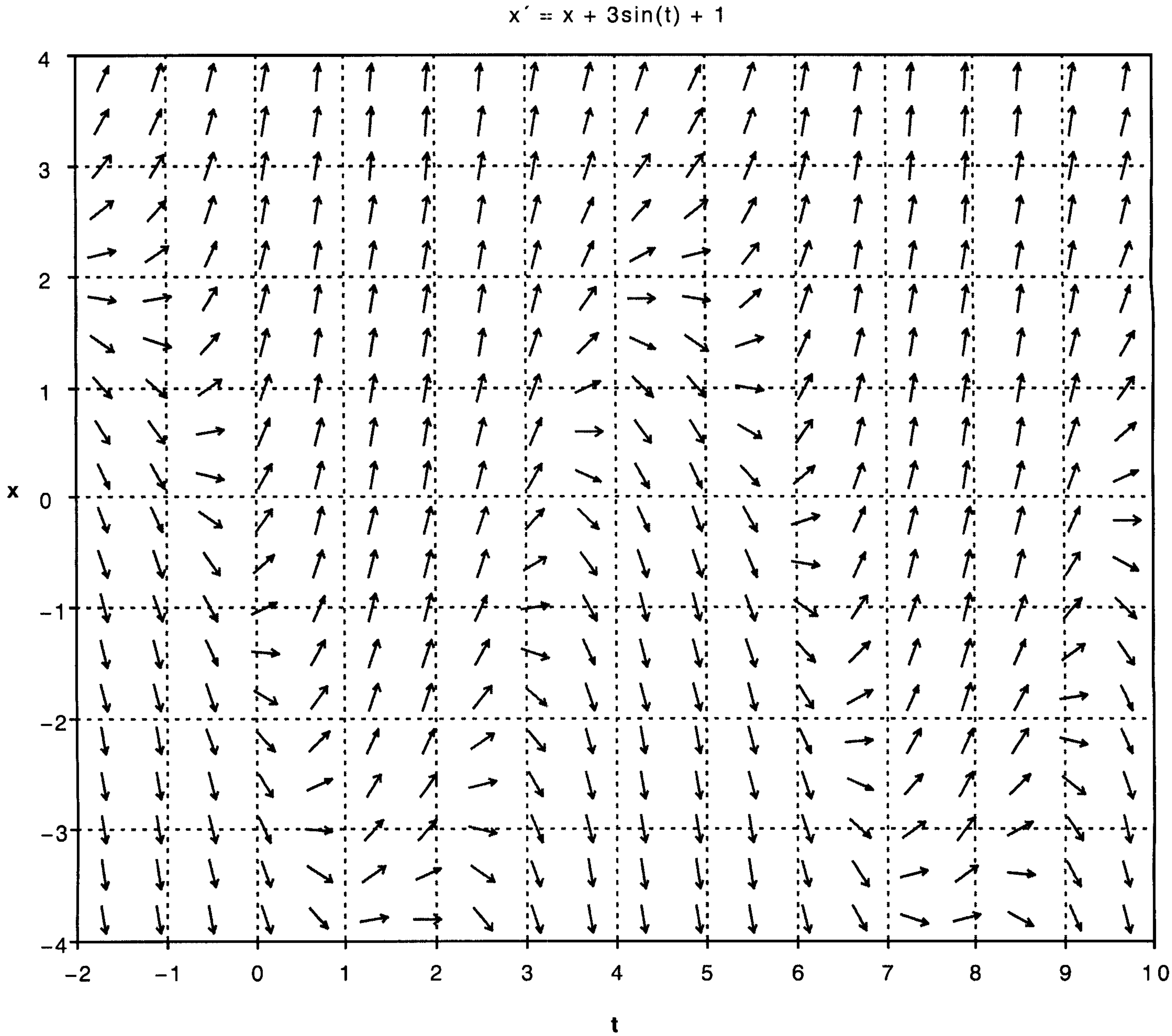
$\int (e^{-t}y)' = \int e^{-t} + \underbrace{e^{-t}3\sin(t)\,dt}$

I.B.P. twice, see HW.

$\Rightarrow y = -1 - \frac{3}{2}(\cos(t) + \sin(t)) + Ce^t.$

for this to remain finite,

$Ce^t \to 0$, or be bounded - only happens if $C = 0$.

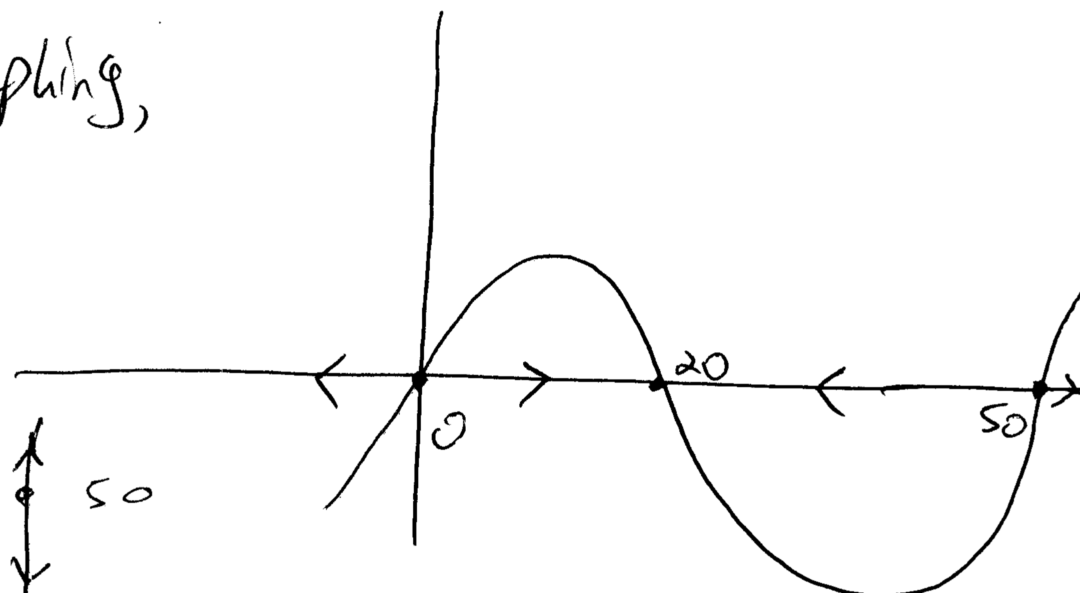


6. a) (10 points) Consider the differential equation

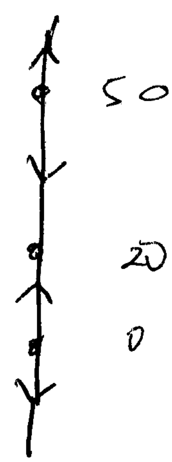
$$y' = (y - 50)(y - 20)y.$$

Identify the equilibrium solutions and classify each as *stable* or *unstable* (drawing a phase diagram might be very, very useful).

graphing,



$\Rightarrow$



$\Rightarrow$ 0, 50 unstable

20 stable

b) (4 points) If the equation above represents the growth of a certain population of chickens that you are raising and you wanted to limit the population so that it does not grow uncontrollably, how many chickens should you start with? (Give a range, i.e. an interval).

want # of chickens to remain bounded, so mathematically, begin with

$0 \leq y_0 \leq 50$ (note the $\leq$ sign)

7. (12 points) Consider the differential equation

$$y' = -ty + 2y^3, \quad y(0) = 1.$$

Use Euler's method to set up the equations with $h = 0.1$ for $t = 0.3$ (you do not need to simplify the solutions).

$y' = f(t,y) = -ty - 2y^3$

$y_1 = y_0 + f(t_0, y_0) \cdot h \qquad (t_0 = 0, \ y_0 = 1)$

$= 1 + (2) \cdot (0.1) = 1.2$

$y_2 = y_1 + f(t_1, y_1) \cdot h \qquad (t_1 = 0.1, \ y_1 = 1.2)$

$= 1.2 + (-(0.1)(1.2) + 2(1.2)^3)(0.1)$

$y_3 = y_2 + f(t_2, y_2) \cdot h$

$= y_2 + f(0.2, y_2) \cdot (0.1)$

$y_4 = y_3 + f(t_3, y_3) \cdot h$

$= y_3 + f(0.3, y_3)(0.1)$