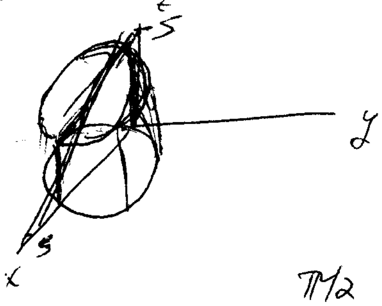
Sample 2

1) a)



$$\Rightarrow \int_{-\pi/2}^{\pi/2}\int_{0}^{3\cos\theta}\int_{0}^{5-r\cos\theta} dz\, r\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_{0}^{3\cos\theta} (5-r\cos\theta)\, r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \frac{5r^2}{2} - \frac{r^3}{3}\cos\theta \Big|_{0}^{3\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{45\cos^2\theta}{2} - 9\cos^4\theta\, d\theta.$$

use $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ twice

$$= \int_{-\pi/2}^{\pi/2} \frac{45}{2}\left(\frac{1+\cos 2\theta}{2}\right) - \frac{9}{4}\left(1 + 2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= \frac{45}{4}\theta + \frac{45}{8}\sin 2\theta + \frac{9}{4}\theta + \frac{9}{4}\sin 2\theta + \frac{9}{8}\theta + \frac{9}{32}\sin 4\theta \Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{45}{4}\pi + \frac{9}{4}\pi + \frac{9}{8}\pi = \boxed{\frac{90}{8}\pi}$$

b) $$\int_{0}^{2\pi}\int_{0}^{\pi}\int_{0}^{1-\cos\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = \int_{0}^{2\pi}\int_{0}^{\pi} \frac{(1-\cos\varphi)^3}{3}\sin\varphi\, d\varphi\, d\theta$$

$u = 1-\cos\varphi$

$du = \sin\varphi\, d\varphi$

$$\Rightarrow \int_{0}^{2\pi} \frac{(1-\cos\varphi)^4}{3\cdot 4}\Big|_{0}^{\pi} d\theta = \frac{2^4}{3\cdot 4}\int_{0}^{2\pi} d\theta = \boxed{\frac{8\pi}{3}}$$

2) $$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$$

2) cont.

bounds	corresponding substitution	simplified
$x = y/2$	$u + v/2 = v/2$	$u = 0$
$x = y/2 + 2$	$u + \frac{v}{2} = \frac{v}{2} + 2$	$u = 2$
$y = 0$	$v = 0$	$v = 0$
$y = 2$	$v = 2$	$v = 2$

$$\Rightarrow \int_0^2 \int_{y/2}^{y/2+2} y^3 (2x-y) e^{(2x-y)^2} \, dx\, dy = \int_0^2\int_0^2 v^3\, 2u\, e^{4u^2} \, du\, dv$$

$$= \frac{1}{4}\int_0^2 v^3 e^{4u^2}\Big|_0^2 dv = \frac{1}{4}\int_0^2 (e^{16}-1)\, v^3\, dv = \frac{v^4}{4}\Big|_0^2 \left(\frac{e^{16}}{4} - \frac{1}{4}\right)$$

$$= \boxed{e^{16}-1}$$

3) $\vec{r}_1(t)$ (for C_1) $= x\vec{i} + y\vec{j} = x\vec{i} + \frac{x^2}{2}\vec{j}$ let $x = t$, $0 \le t \le 2$

$\Rightarrow \vec{r}_1(t) = t\vec{i} + \frac{t^2}{2}\vec{j}$ $\Rightarrow \vec{r}'(t) = \vec{i} + t\vec{j}$

$\Rightarrow |\vec{r}'(t)| = \sqrt{1+t^2}$

$$\Rightarrow \int_{C_1} = \int_0^2 \frac{t^3}{t^2/2}\sqrt{1+t^2}\, dt = \int_0^2 2t\sqrt{1+t^2}\, dt = \frac{2}{3}(1+t^2)^{3/2}\Big|_0^2$$

$u = 1+t^2$

$du = 2t$

$$= \boxed{\frac{2}{3}\left(5^{3/2}-1\right)}$$

$\vec{r}_2(t)$: (for C_2), $(2,2)$ to $(4,3)$

$x(t) = 2 + 2t$ $\quad 0 \le t \le 1.$ $\quad |r'(t)| = \sqrt{1^2+2^2} = \sqrt{5}$

$y(t) = 2 + t$

$$\Rightarrow \int_{C_2} = \int_0^1 \frac{(2+2t)^3}{2+t}(\sqrt{5})\, dt = \int_0^1 (\sqrt{5})\frac{(8 + 8t + 8t^2 + 8t^3)}{2+t}\, dt$$

using $\frac{x}{2+x} = 1 - \frac{2}{2+x}$

$$= \sqrt{5}\left(8\ln(2+t) + 8t - 16\ln(2+t) + \frac{8t^2}{2} - 16t + 32\ln(2+t) + \frac{8t^3}{3} - \frac{16t^2}{2} + 32t - 64\ln(2+t)\right)\Big|_0^1 = \left(\begin{matrix}-40\ln(3)\\+40\ln(2)\end{matrix} + 8 + 4 - 16 + \frac{8}{3} - 8 + 32\right)\sqrt{5}$$

$$= \boxed{40 \ln \frac{2}{3} + \frac{68}{3}}$$

$$\Rightarrow \text{total} = \boxed{\frac{2}{3}\left(5^{3/2} - 1\right) + 40 \ln \frac{2}{3} + \frac{68}{3}}$$

4)

(z should have been ≥ 0).

$\delta(x,y,z) = 2 - z$, $\vec{r}(t) = \cos t\,\vec{j} + \sin t\,\vec{k}$ $\quad 0 \leq t \leq \pi$

$|r'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1. \Rightarrow M = \int_C \delta\, ds = \int_C (2 - z)\, ds$

$= \int_0^{\pi} (2 - \sin t)\, 1\, dt = 2\pi - 2.$

$\Rightarrow I_x = \int_C (y^2 + z^2)\,\delta\, ds = \int_0^{\pi} (\cos^2 t + \sin^2 t)(2 - \sin t)\, dt = \int_0^{\pi} 2 - \sin t\, dt = 2\pi - 2.$

$\Rightarrow R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{2\pi - 2}{2\pi - 2}} = \boxed{1}.$

5) $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

$\Rightarrow w = \int_C \vec{F} \cdot \vec{T}\, ds = \int_{t_0}^{t_f} F \cdot \underbrace{\frac{d\vec{r}}{dt}}_{= \vec{r}'(t)}$ $\qquad \vec{F} = (t^2 + t^4)\vec{i} + (t^4 + t)\vec{j} + (t + t^2)\vec{k}$

$\frac{d\vec{r}}{dt} = \vec{i} + 2t\vec{j} + 4t^3 k$

$\Rightarrow \vec{F} \cdot \vec{r}'(t) = 6t^5 + 5t^4 + 3t^2 \quad \Rightarrow w = \int_0^1 6t^5 + 5t^4 + 3t^2\, dt = t^6 + t^5 + t^3 \Big|_0^1$

$= \boxed{3}.$

6) Circulation

$\vec{r}_1(t) = 3\cos t\,\vec{i} + 3\cos t\,\vec{j} \quad r_1'(t) = -3\sin t\,\vec{i} + 3\cos t\,\vec{j} \quad 0 \leq t \leq \pi$

$\vec{r}_2(t) = t\vec{i} + 0\vec{j}$, $-3 \leq t \leq 3$ (i.e. $t = x$). $\quad r_2'(t) = 1\,\vec{i}$

$\Rightarrow \text{Circulation} = \int_C \vec{F} \cdot \vec{T}\, ds =$

$= \int_0^{\pi} (-3\sin t\,\vec{i} + 3\cos t\,\vec{j})(-3\sin t\,\vec{i} + 3\cos t\,\vec{j})\, dt = \int_0^{\pi} 9\, dt = 9\pi$

$+ \int_{-3}^{3} (0\,\vec{i} + t\vec{j})(1\,\vec{i} + 0\vec{j})\, dt = 0 \qquad \Rightarrow \text{Circulation} = \boxed{9\pi}$

$\text{Flux} = \int_C \vec{F} \cdot \vec{n}\, ds = \int_{C_1} M_1\, dy - N_1\, dx + \int_{C_2} M_2\, dy - N_2\, dx = \cancel{\int_0^{\pi} (-3\sin t)(3\cos t) - (3\cos t)(-3\sin t)\, dt}\ 0$

$+ \int_{-3}^{3} 0 \cdot 0 - t\, dt = -\frac{t^2}{2}\Big|_{-3}^{3} = 0. \Rightarrow \text{Flux} = \boxed{0}$