Sample 2 Key

1) a) $f(x,y) = \sqrt{4 - x^2 - 2y^2 + x}$ $\Rightarrow$ domain $(4 - x^2 - 2y^2 - x) \geq 0$

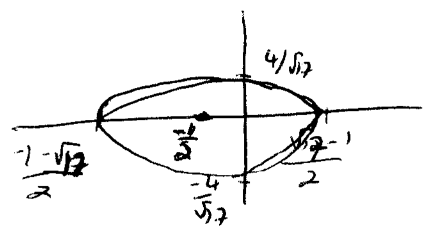
$$\Rightarrow 4 \geq x^2 + x + 2y^2$$

$$\Rightarrow 4 \geq x^2 + x + \left(\frac{1}{2}\right)^2 + 2y^2 - \left(\frac{1}{2}\right)^2$$

$$\Rightarrow 4 + \left(\frac{1}{2}\right)^2 \geq \left(x + \frac{1}{2}\right)^2 + 2y^2$$

So the domain is an ellipse centered at $\left(\frac{-1}{2}, 0\right)$ and everything inside it.

$$4 + \frac{1}{4} = \frac{17}{4} \Rightarrow \left(\frac{\sqrt{17}}{2}\right)^2 \geq \left(x + \frac{1}{2}\right)^2 + 2y^2$$



range $\Rightarrow z \in [0, 2]$, that is $0 \leq z \leq 2$, because x and y are points on or in the ellipse, any point on the ellipse means $x^2 + x + 2y^2 = 4 \Rightarrow z = 0$, and in, the smallest that it could be is $x + y = 0 \Rightarrow \sqrt{4} = 2$. Note that the function doesn't specify positive or negative values of the square root, so it's ok to assume only positive.

b) $F(x, y, z) = \dfrac{z^2}{x + y}$ so the denominator cannot equal 0,

$$\Rightarrow x + y \neq 0 \Rightarrow x \neq -y.$$

So the domain is: x - all reals
y - all reals - as long as $x \neq -y$,
z - all reals.

The range is all reals in our case.

c) $F(x,y) = \ln\left(1 - \dfrac{1-x}{1-y}\right)$ so for the domain, $\ln(\square)$ must be > 0.

$$\Rightarrow 1 - \frac{1-x}{1-y} > 0 \Rightarrow 1 > \left(\frac{1-x}{1-y}\right)$$

$$\Rightarrow 1 - y > x - 1 \Rightarrow -y > x - 2 \Rightarrow \boxed{y < 2 - x}$$

Also though, $\frac{1-x}{1-y}$ – $1-y \neq 0!$ $\Rightarrow y \neq 1.$

so our domain is $y < 2-x$ and $y \neq 1.$

The range of $\ln$ is all real numbers.

2) a) $f(x,y) = xy^3 + (x-y)^2 + 2$

$f_x = y^3 + 2(x-y)$ $\quad f_{xx} = 2$ $\quad f_{xy} = 3y^2 - 2$

$f_y = 3xy^2 - 2(x-y)$ $\quad f_{yy} = 6xy + 2$

↑ is $f_{yx} = f_{xy}$? check...

b) $f(x,y) = x e^{xy-y-1}$

f_x = (Product rule) = $e^{xy-y-1} + x e^{xy-y-1} \cdot (y)$

$f_{xx} = y e^{xy-y-1} + y e^{xy-y-1} + xy^2 e^{xy-y-1}$

$= 2y e^{xy-y-1} + xy^2 e^{xy-y-1}$

$f_{xy} = (x-1) e^{xy-y-1} + x e^{xy-y-1} + xy(x-1) e^{xy-y-1}$

$f_y = x(x-1) e^{xy-y-1}$

$f_{yy} = x(x-1)^2 e^{xy-y-1}.$

c) $f(x,y) = \ln\left(x - \frac{12}{y}\right)$

$f_x = \dfrac{1}{x - \frac{12}{y}}$ $\quad f_{xx} = \dfrac{-1}{\left(x - \frac{12}{y}\right)^2}$

f_{xy} - 1st rewrite f_x $\quad \dfrac{1}{x - \frac{12}{y}} \left(\dfrac{y}{y}\right) = \dfrac{y}{xy-12}$

$\Rightarrow f_{xy} = \dfrac{(xy-12) - yx}{(xy-12)^2} = \dfrac{-12}{(xy-12)^2}$

$$f_y = \frac{\frac{-12}{y^2}}{x - \frac{12}{y}} = \frac{-12}{y^2\left(x - \frac{12}{y}\right)} = \frac{-12}{y^2 x - 12y}$$

$$f_{yy} = \frac{12(2yx - 12)}{y^2 x - 12y}$$ (Quotient rule low • d high − high • d low, square the bottom and here we go! ...)

3) a) $f(x,y) = x^2 + 2xy + y^2$.

(1) $f_x = 2x + 2y$ $\quad f_{xx} = 2, \quad f_{xy} = 2$ ✓

$f_y = 2x + 2y$ $\quad f_{yy} = 2 \quad f_{yx} = 2$

(2) critical points $\begin{matrix} f_x = 0 \\ f_y = 0 \end{matrix} \Rightarrow \begin{matrix} 2x + 2y = 0 \\ 2x + 2y = 0 \end{matrix} \Rightarrow x = -y.$

(3) classify $d = f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$

$= 2 \cdot 2 - 2^2 = 0!$ — no info!

So let's plug in. – what happens when $x = -y$?

$$x^2 + 2x(-x) + (-x)^2 = x^2 - 2x^2 + x^2 = \underline{0}!$$

But this is a critical point – so let's compare, try $x = y$ (or to be succinct – you can try $x = y = 1$)

$\Rightarrow$ ~~...~~ $1^2 + 2 \cdot (1)(1) + (1)^2 = 4 > 0!$ ~~So our guess~~

~~...~~ Any other point w/ $x \neq -y$ will also give a positive number (try it!)

So our guess is that at $x = -y$, $f(x,y)$ has a line of absolute minimum. if you look at $f(x,y) = x^2 + 2xy + y^2$, and notice that $f(x,y) = (x+y)^2$, this would make sense since $(x+y)^2 \geq 0$ always!

3 b) $f(x,y) = x + y^2 - e^x$

(1) $f_x = 1 - e^x$ $\quad f_{xx} = -e^x$ $\quad f_{xy} = 0$

$f_y = 2y$ $\quad f_{yy} = 2$ $\quad f_{yx} = 0$

(2) $\Rightarrow$ $f_x = 0$ $\quad 1 - e^x = 0 \Rightarrow 1 = e^x \Rightarrow x = 0$

$f_y = 0$ $\quad 2y = 0 \Rightarrow y = 0.$

(3) $d = f_{xx}(x_0,y_0)\, f_{yy}(x_0,y_0) - (f_{xy}(x_0,y_0))^2$

$= (-e^x) \cdot 2 - (0)^2 = -2e^x$ at $(0,0)$

So $d < 0$ $\quad = -2.$

$\Rightarrow$ saddle point.

c) $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 8$

(1) $f_x = 3x^2 + 3y^2 - 6x$ $\quad f_{xx} = 6x - 6$ $\quad f_{xy} = 6y.$

$f_y = 6xy - 6y$ $\quad f_{yy} = 6x - 6$ $\quad f_{yx} = 6y$

(2) $f_x = 0$ $\quad 3x^2 + 3y^2 - 6x = 0 \Rightarrow 3(x^2 + y^2 - 2x) = 0$

$f_y = 0$ $\quad 6xy - 6y = 0 \Rightarrow 6y(x-1) = 0. \Rightarrow x = 1$ or $y = 0.$

If $x = 1 \Rightarrow x^2 + y^2 - 2x = 1 + y^2 - 2 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1.$

so $(1,1)$ & $(1,-1)$

if $y = 0 \Rightarrow x^2 + y^2 - 2x = x^2 - 2x = 0 \Rightarrow x(x-2) = 0$ $\quad x = 0$ or $x = 2$

so $(0,0)$ & $(2,0).$

So we have 4 critical points, $(1,1), (1,-1), (0,0), (2,0)$

(3) $D = f_{xx}(x_0,y_0) \cdot f_{yy}(x_0,y_0) - \left(f_{xy}(x_0,y_0)\right)^2$

$= (6x-6)(6x-6) - (6y)^2$

at $(1,1)$: $(6-6)^2 - (6)^2 = -36$ $\quad D<0 \Rightarrow$ $\boxed{\text{Saddle point at } (1,1)}$

at $(1,-1)$: $(6-6)^2 - (-6)^2 = -36$, $D<0 \Rightarrow$ $\boxed{\text{Saddle point at } (1,-1)}$

at $(0,0)$: $(-6)^2 = 36$, $D>0$.

$f_{xx} = 6\cdot 0 - 6 = -6 < 0 \Rightarrow$ $\boxed{\text{maximum at } (0,0)}$

at $(2,0)$: $(12-6)^2 - 0^2 = 36 > 0$.

$f_{xx} = 6\cdot 2 - 6 = 6 > 0 \Rightarrow$ $\boxed{\text{minimum at } (2,0)}$

(4) $x+y+z=8$, maximize $x\cdot y\cdot z$.

(can think of this as the volume of a box with a vertex on the plane of $x+y+z=8$!).

$\Rightarrow z = 8-x-y$, $\Rightarrow V = (xy)(8-x-y) = 8xy - x^2y - y^2x$.

$f_x = 8y - 2xy - y^2$ $\quad f_{xx} = -2y$ $\quad f_{xy} = 8 - 2x - 2y$

$f_y = 8x - x^2 - 2yx$ $\quad f_{yy} = -2x$ $\quad f_{yx} = 8 - 2x - 2y$

$f_x = 0 \Rightarrow 8y - 2xy - y^2 = 0 \Rightarrow y(8-2x-y) = 0 \Rightarrow y=0$ or $8-2x-y=0$

$f_y = 0 \Rightarrow 8x - x^2 - 2yx = 0 \Rightarrow x(8-2y-x) = 0. \Rightarrow x=0$ or $8-2y-x=0$.

So $x=0, y=0$ - one point. $(0,0)$

$x=0$, $8-2x-y=0 \Rightarrow 8-y=0 \Rightarrow y=8$ $\quad (0,8)$

$8-2y-x=0$ & $y=0 \Rightarrow x=8$, $(8,0)$

$8-2y-x=0$ & $8-2x-y=0$ $\Rightarrow$ $8-2y=x$, substitute into $8-2x-y=0$.

$\Rightarrow 8-2(8-2y)-y=0$

$8-16+4y-y=0 \Rightarrow 3y=8 \Rightarrow y=8/3.$

So $8-2y=x \Rightarrow 8-2\left(\frac{8}{3}\right) = \frac{24}{3}-\frac{16}{3} = \frac{8}{3} = x$

$\Rightarrow \left(\frac{8}{3},\frac{8}{3}\right).$

By inspection, you can see that $(0,0)$, $(8,0)$ & $(0,8)$ will all give us a volume of 0. So $\left(\frac{8}{3},\frac{8}{3}\right)$ must be the max.

Checking: $D = f_{xx}(x_0,y_0) f_{yy}(x_0,y_0) - \left(f_{xy}(x_0,y_0)\right)^2$

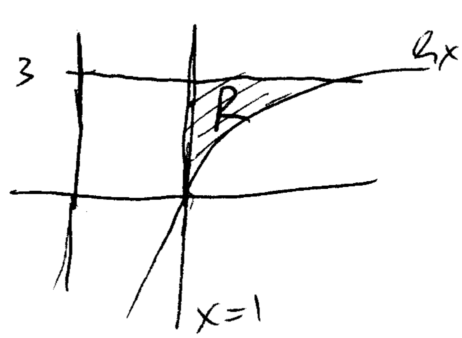
$= (-2y)(-2x) - (8-2x-2y)^2$

$= 4\left(\frac{8}{3}\right)^2 - \left(8-2\left(\frac{8}{3}\right)-2\left(\frac{8}{3}\right)\right)^2$

$= 4\left(\frac{64}{9}\right) - \left(\frac{8}{3}\right)^2 > 0$

$f_{xx} = -2y = -2\left(\frac{8}{3}\right) < 0 \Rightarrow \boxed{\left(\frac{8}{3},\frac{8}{3}\right) \text{ is a maximum.}}$

5) Check solution Manual.

6) 

note that $\ln x$ starts being positive at $x=1$.

$\Rightarrow$ horizontal R: $1 \le x \le e^y$, $0 \le y \le 3$

vertical R^*: $\ln x \le y \le 3$, $1 \le x \le e^3$

6) cont' Area of R

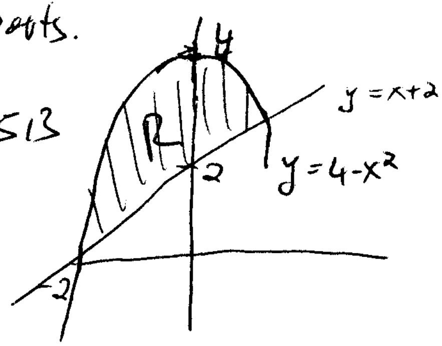
$$\int_0^3 \int_1^{e^y} dx\,dy \quad \text{is easier,}$$

$$= \int_0^3 (e^y - 1)\,dy = e^y - y\Big|_0^3 = e^3 - 3 - e^0$$

$$\boxed{= e^3 - 4}$$

if you integrated wrt y then x, you would need to use Integration by parts.

7) #38 p. 513



R - horizontal - at parts will go from $y = 4-x^2$ to $y = x+2$, & at times from $y = 4-x^2$ at one end, to $y = 4-x^2$ at the other end - not a good idea.

R^* - vertical $\quad x+2 \le y \le 4-x^2$

get these by setting $y = x+2$ equal to $y = 4-x^2$ $\quad -2 \le x \le 1$

$$\Rightarrow \quad x+2 = 4-x^2 \Rightarrow x^2 + x - 2 = 0 \Rightarrow (x+2)(x-1) = 0.$$

$$\Rightarrow \int_{-2}^{1} \int_{x+2}^{4-x^2} dy\,dx = \int_{-2}^{1} (4 - x^2 - x - 2)\,dx$$

$$= 4x - \frac{x^3}{3} - \frac{x^2}{2} - 2x \Big|_{-2}^{1}$$

$$= 4 - \frac{1}{3} - \frac{1}{2} - 2 - \left(4(-2) - \frac{(-2)^3}{3} - \frac{(-2)^2}{2} - 2(-2)\right)$$

$$= 4 + 8 + 2 - \frac{1}{3} - \frac{1}{2} - 2 - \frac{8}{3} - 4 = 4\tfrac{1}{2} = \boxed{9/2}$$

12. $\displaystyle\int_0^4\int_0^{\sqrt{x}} \frac{y}{1+x^2}\,dy\,dx = \int_0^2\int_{y^2}^4 \frac{y}{1+x^2}\,dx\,dy$

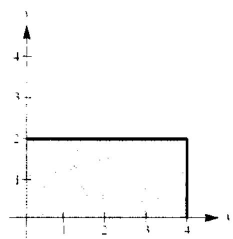
$$\int_0^4\int_0^{\sqrt{x}} \frac{y}{1+x^2}\,dy\,dx = \int_0^4 \left[\frac{y^2}{2(1+x^2)}\right]_0^{\sqrt{x}} dx$$
$$= \int_0^4 \frac{x}{2(1+x^2)}\,dx$$
$$= \frac{1}{4}\ln(1+x^2)\Big]_0^4$$
$$= \frac{1}{4}\ln 17$$
$$\approx 0.708$$

14. $\displaystyle\int_0^{\ln 10}\int_{e^x}^{10} \frac{1}{\ln y}\,dy\,dx = \int_1^{10}\int_0^{\ln y} \frac{1}{\ln y}\,dx\,dy$

$$= \int_1^{10}\left[\frac{x}{\ln y}\right]_0^{\ln y} dy = \int_1^{10} dy = 9$$

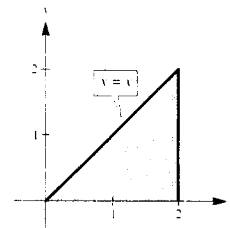
16. $\displaystyle V = \int_0^4\int_0^2 (6-2y)\,dy\,dx$

$$= \int_0^4 \Big[6y - y^2\Big]_0^2 dx$$
$$= \int_0^4 8\,dx = 32$$



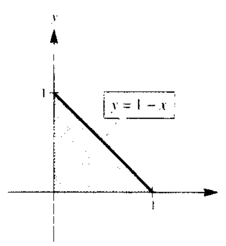
18. $\displaystyle V = \int_0^2\int_0^x 6\,dy\,dx$

$$= \int_0^2 6x\,dx$$
$$= 3x^2\Big]_0^2 = 12$$



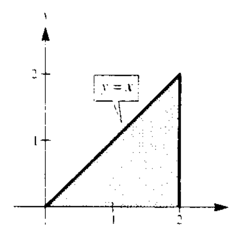
20. $\displaystyle V = \int_0^1\int_0^{1-x} (1-x-y)\,dy\,dx$

$$= \int_0^1 \left[(1-x)y - \frac{y^2}{2}\right]_0^{1-x} dx$$
$$= \int_0^1 \frac{1}{2}(1-x)^2\,dx$$
$$= -\frac{1}{6}(1-x)^3\Big]_0^1 = \frac{1}{6}$$



22. $\displaystyle V = \int_0^2\int_0^y (4-y^2)\,dx\,dy$

$$= \int_0^2 y(4-y^2)\,dy$$
$$= \left[2y^2 - \frac{y^4}{4}\right]_0^2 = 4$$



24. $\displaystyle\int_0^\infty\int_0^\infty \frac{1}{(x+1)^2(y+1)^2}\,dx\,dy = \int_0^\infty \lim_{b\to\infty}\left[\frac{-1}{(x+1)(y+1)^2}\right]_0^b dy$

$$= \int_0^\infty \frac{1}{(y+1)^2}\,dy$$
$$= \lim_{b\to\infty}\left[-\frac{1}{y+1}\right]_0^b = 1$$

26. $\displaystyle V = \int_0^\infty\int_0^\infty e^{-(x+y)/2}\,dy\,dx$

$$= \int_0^\infty \lim_{b\to\infty}\left[-2e^{-(x+y)/2}\right]_0^b dx$$
$$= \int_0^\infty 2e^{-x/2}\,dx$$
$$= \lim_{b\to\infty}\left[-4e^{-x/2}\right]_0^b = 4$$

28. $\displaystyle V = \int_0^4\int_0^x x\,dy\,dx = \int_0^4 x^2\,dx = \frac{x^3}{3}\Big]_0^4 = \frac{64}{3}$

✓ extras...

9.) ___Note:___ There is a mistake in #9, $f(x,y)$ should be $\frac{y}{x}$ not $\frac{x}{y}$. $\frac{x}{y}$ is a much more difficult integral, and that with the bounds in #9 will not be on the exam.

So from #6 $\underbrace{\frac{1}{A}\int_0^3\int_1^{e^y} \frac{y}{x}\,dx\,dy}_{\text{easier!}}$ or $\frac{1}{A}\int_1^{e^3}\int_{\ln x}^{3} \frac{y}{x}\,dy\,dx$

$$\Rightarrow \frac{1}{e^3-4}\left(\int_0^3 \left(y\ln x\Big|_1^{e^y}\right)dy\right) = \frac{1}{e^3-4}\left(\int_0^3 (y\ln e^y - y\ln 1)\,dy\right)$$

$$= \frac{1}{e^3-4}\int_0^3 y^2\,dy = \frac{1}{e^3-4}\,\frac{y^3}{3}\Big|_0^3 = \frac{1}{e^3-4}\,\frac{27}{3} = \boxed{\frac{9}{e^3-4}}$$

is the average value.