Sample 3 Solutions

1) a) $n=1 \quad n=2 \quad 3 \quad 4 \quad 5$

$1 \quad -3 \quad 7 \quad -15 \quad 31$ — sign alternates, with negative on the even n's $\Rightarrow (-1)^{n+1}$

$+2 \quad +4 \quad +8 \quad +16$ (in absolute value of a_n) $1 \to 3 \to 7 \to 15 \to 31 \ldots$

Clearly there is a relation with 2. But notice, each a_n is one less than a power of 2!

$$\Rightarrow \quad a_n = (2^n - 1)(-1)^{n+1} \qquad \lim_{n\to\infty} (2^n-1)(-1)^{n+1} \to \pm\infty$$

$\boxed{DNE}$

b) $n=1 \quad 2 \quad 3 \quad 4$

$1 \quad \frac{3}{5} \quad \frac{3}{7} \quad \frac{1}{3}$

Denominator: $1 \quad 5 \quad 7 \quad 3$

$+4 \quad +2 \quad -4$ — doesn't seem like much of a connection (remember, we want to be able to write this as a mathematical formula)

but note: $1 \cdot \frac{3}{5} = \frac{3}{5}$, $\frac{3}{5} \cdot \frac{5}{7} = \frac{3}{7}$, $\frac{3}{7} \cdot \frac{7}{9} = \frac{3}{9} = \frac{1}{3}$!

So there is a connection here — there is a 3 on each numerator $\Rightarrow$ $1 = \frac{3}{3}, \frac{3}{5}, \frac{3}{7}, \frac{1}{3} = \frac{3}{9}$.

$$\Rightarrow \quad a_n = \frac{3}{2n+1}, \quad \lim_{n\to\infty}\left(\frac{3}{2n+1}\right) = 0 \quad \text{— converges.}$$

c) $3, \frac{6}{2}, \frac{11}{6}, \frac{18}{24}, \frac{27}{120} \ldots$

Denominator: $1 \quad 2 \quad 6 \quad 24 \quad 120 \quad \to (n!)$

numerator $3 \quad 6 \quad 11 \quad 18 \quad 27$

$+3 \quad +5 \quad +7 \quad +9$

Note that for $a_n = n^2$,

$$\begin{array}{cccccc} 1 & 4 & 9 & 16 & 25 & 36 \end{array}$$
$$+3 \quad +5 \quad +7 \quad +9 \quad +11$$

$\Rightarrow$ some connection to n^2! $\Rightarrow$ n^2+2!

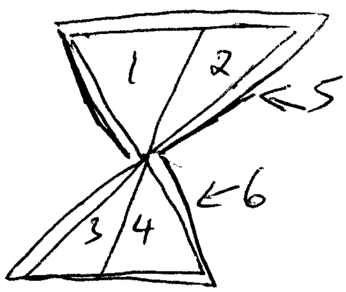
So $a_n = \dfrac{n^2+2}{n!}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n^2+2}{n!}\right) \longrightarrow 0$$

(to see this you need to realize that as n gets large, 2 doesn't matter,

$$\Rightarrow \sim \frac{n^2}{n!} = \frac{n}{(n-1)!} = \frac{n}{\underbrace{(n-1)(n-2)(n-3)\cdots}_{\text{much bigger than } n}}$$
)

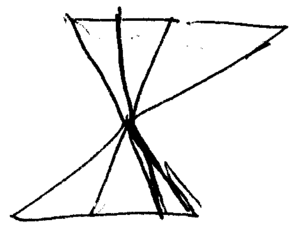
Challenge:

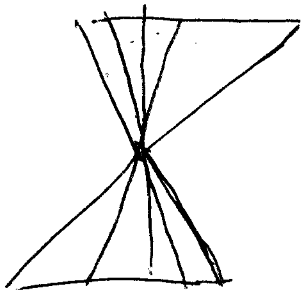


$n=0$ $-$ 2 triangles.

$n=1$ $-$ $a_1 = 6$

$n=2$ $\quad a_2 = 12$ (count carefully!)





$n=3$ $\quad a_3 = 20$

$$\Rightarrow \quad a_1 = 6 = 2\cdot 3$$
$$a_2 = 12 = 3\cdot 4$$
$$a_3 = 20 = 4\cdot 5$$
$$a_n = (n+1)(n+2)$$
$$\Rightarrow \quad a_{200} = \boxed{(201)(202)}\ !$$

2) a) $\sum_{n=1}^{\infty} \frac{n^2+1}{n(n+1)}$ note - same power on top & on bottom

$\Rightarrow$ n^{th} term test!

$$\lim_{n\to\infty} \frac{n^2+1}{n^2+n} = \lim_{n\to\infty} \frac{(n^2+1)\ 1/n^2}{(n^2+n)\ 1/n^2} = \lim_{n\to\infty} \frac{1+1/n^2}{1+1/n} = 1 \neq 0!$$

$\Rightarrow$ $\boxed{\text{Diverges. by } n^{th} \text{ term test}}$

b) $\sum_{i=-1}^{20} i$ - this is a finite sum (only goes up to 20)

$\Rightarrow$ will converge!

$= -1 + 0 + 1 + 2 + 3 + 4 + 5 + \dots + 20$

$= -1 + \frac{(20)(20+1)}{2}$ (using the formula $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$)

$= -1 + 10 \cdot 21 = \boxed{209}$

The key thing to take from here is that this is a finite sum

$\Rightarrow$ will converge.

c) $\sum_{n=1}^{\infty} \frac{15n^3 - 2n^6 + 1001}{n^6 - 42n^2 + \pi n^2 - 4}$ powers of n on top & bottom

$\Rightarrow$ try n^{th} term test 1st.

$\Rightarrow$ largest power of n in the denominator is 6.

$$\Rightarrow \lim_{n\to\infty} \frac{15n^3 - 2n^6 + 1001}{n^6 - 42n^2 - \pi n^2 - 4} \cdot \frac{1/n^6}{1/n^6} = \lim_{n\to\infty} \frac{15/n^3 - 2 + \frac{1001}{n^6}}{1 - \frac{42}{n^4} - \frac{\pi}{n^4} - \frac{4}{n^4}} = -2 \neq 0$$

$\Rightarrow$ n^{th} term test $\neq 0$ - $\boxed{\text{diverges}}$!
$\boxed{\text{by } n^{th} \text{ term test}}$

d) $\sum_{k=0}^{\infty} 6\left(\frac{-4}{3}\right)^k$ - geometric series!

$a = 6$, $r = -\frac{4}{3}$.

$|r| = \left|-\frac{4}{3}\right| > 1 \Rightarrow$ $\boxed{\text{diverges by geometric series } |r| > 1}$

e) $\sum_{n=0}^{\infty}\left[(0.5)^n + (0.2)^n\right] = \underbrace{\sum_{n=0}^{\infty}(0.5)^n + \sum_{n=0}^{\infty}(0.2)^n}$

both <1 in absolute value, so series will converge $\Rightarrow$ can do this. (separate them)

$$\Rightarrow \sum_{n=0}^{\infty}(0.5)^n = \frac{1}{1-0.5} = 2$$

$$\sum_{n=0}^{\infty}(0.2)^n = \frac{1}{1-0.2} = \frac{1}{1-\frac{1}{5}} = \frac{5}{4}.$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[(0.5)^n + (0.2)^n\right] = 2 + \frac{5}{4} = \boxed{\frac{13}{4} \text{ - converges, geometric series}}$$

Note that this is not the same as $\sum_{n=0}^{\infty}(0.7)^n = \frac{10}{7}$.

f) $\sum_{n=5}^{72} 4\left(\frac{-1}{4}\right)^n$ – another finite sum, of the geometric variety.

$a = 4$
$r = -\frac{1}{4}$ $\Rightarrow$ $\frac{ar^k - ar^{n+1}}{1-r}$ (☆)

1st term in sum

(last term)$\cdot r$

$$= \frac{4\left(\frac{-1}{4}\right)^5 - 4\left(\frac{-1}{4}\right)^{72+1}}{1-\left(\frac{-1}{4}\right)} = \boxed{\frac{4\left(\frac{-1}{4}\right)^5 - 4\left(\frac{-1}{4}\right)^{73}}{5/4}}.$$

Note that we can do this for $|r|>1$, as long as it's a finite sum – i.e. $\sum_{n=5}^{72}(4)^n = \frac{4^5 - 4^{73}}{1-4}$, in order to see this remember how we got to the formula above (or (☆))

$\Rightarrow \sum_{n=5}^{k} r^n = r^5 + r^6 + \dots + r^k$, multiply by $(1-r)$

$\Rightarrow r^5 + r^6 + \dots + r^k - r^6 - r^7 - \dots - r^k - r^{k+1} = r^5 - r^{k+1}$ ↪

$$r^5 - r^{k+1} = \left(\sum_{n=5}^{k} r^n\right)(1-r) \Rightarrow \sum_{n=5}^{k} r^n = \frac{r^5 - r^{k+1}}{1-r}.$$

g) $0.18 + 0.0018 + 0.000018 + \cdots$ (dividing term n by 100 to get to term $n+1$).

$= \sum_{n=0}^{\infty} (.18)(.01)^n$ — geometric

$a = .18$

$r = .01$; starting at $n=0$. $|r| = 0.01 < 1$, converges.

$\Rightarrow$ sum is $\frac{ar^0}{1-r}$ ← 1st term $= \frac{.18 \cdot 1}{1-.01} = \frac{18/100}{99/100}$

a cubist painting

$= \frac{18}{99} = \boxed{\frac{2}{11}}$ converges, a geometric series.

h) - look in solution manual.

3) a) $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ — a p-series, $\boxed{p=1 \text{ - diverges!}}$

this is $\frac{1}{2}$ times the Harmonic series!

b) $\sum_{n=5}^{\infty} \frac{n 4^n}{n!}$ — exponential 4^n + factorial! ⇒ use ratio test.

$$\Rightarrow \lim_{n\to\infty} \left| \frac{(n+1)\,4^{n+1}}{(n+1)!} \cdot \frac{n!}{n\, 4^n} \right| = \lim_{n\to\infty} \left| \frac{4(n+1)}{(n+1)\,n} \right| = \lim_{n\to\infty} \left| \frac{4}{n} \right| = 0.$$

$\Rightarrow$ $\boxed{\text{Converges by ratio test.}}$

Note that the fact that the sum starts at $n=5$ doesn't make a difference. We only care about what happens at the end of the sum - that is does the n^{th} term go to zero, or does the ratio of the $(n+1)^{th}$ term to the n^{th} term exceed 1, etc. The beginning few terms won't affect convergence or divergence!

c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{\sqrt[3]{n^{10}}}$ — powers of n — p-series!

rewrite as $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^{10/3}} = \sum_{n=1}^{\infty} \frac{1}{n^{10/3} \cdot n^{-2}} = \sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$. $4/3 > 1$

$\Rightarrow$ $\boxed{\text{converges! } p\text{-series}}$

d) $\displaystyle\sum_{n=3}^{\infty} \frac{2n}{1-4^n}$ — a 4^n on the bottom — ratio test!

$$\Rightarrow \lim_{n\to\infty} \left| \frac{\cancel{2}(n+1)}{1-4^{n+1}} \cdot \frac{1-4^n}{\cancel{2}n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)}{n} \cdot \frac{1-4^n}{1-4^{n+1}} \right|$$

Here you have to realize 2 things:

1. $\frac{n+1}{n} \to 1$ as $n \to \infty$.
2. For large n, $\frac{1-4^n}{1-4^{n+1}}$ is almost the same as $\frac{-4^n}{-4^{n+1}}$ — that is for $\frac{-4^n+1}{-4^{n+1}+1}$, the $+1$ doesn't really matter as -4^n is very negative, as is -4^{n+1}.

$\Rightarrow$ Can also see this by multiplying everything by $\frac{1}{4^{n+1}}$

$$\Rightarrow \lim_{n\to\infty} \left| \left(\frac{(n+1)}{(n)}\right) \cdot \left(\frac{\frac{1}{4^{n+1}} - \frac{1}{4}}{\frac{1}{4^{n+1}} - 1} \right) \right| = \left| \frac{-1/4}{-1} \right| = \frac{1}{4} < 1$$

$\Rightarrow$ $\boxed{\text{Converges by ratio test.}}$

e) $\sum_{n=1}^{\infty}\left(5n! + \frac{10}{(n+2)!}\right)$ factorials - ~~ra~~ ratio test - right?

Look closely. $\sum_{n=1}^{\infty}\frac{10}{(n+2)!}$ - will converge, no dependence on n in the numerator - use ratio test to see this.

If $\sum_{n=1}^{\infty} 5n!$ converges, we can separate this into 2 sums and work accordingly. If it diverges, then ~~[illegible]~~ $\sum_{n=1}^{\infty}\frac{10}{(n+2)!}$ ~~[illegible]~~, converging does not effect it (in a sense, it's already at "∞", adding a small amount to it will not move it away from "∞".).

Now, $\sum_{n=1}^{\infty} 5n!$ intuitively diverges, it gets bigger and bigger!

Try the n^{th} term test or the ratio test to see this. $\Rightarrow$ $\boxed{\sum_{n=1}^{\infty}\left(5n! + \frac{10}{(n+2)!}\right) \text{ diverges!}}$

Can also see this by doing the n^{th} term test right away, $\lim_{n\to\infty}\left(5n! + \frac{10}{(n+2)!}\right) \longrightarrow \infty$! $\Rightarrow$ $\boxed{\text{diverges by } n^{th} \text{ term test.}}$

4) a) For Radius of Convergence (R.O.C.) use the ratio test.

$$\Rightarrow \sum_{n=1}^{\infty}(2x)^n \Rightarrow \lim_{n\to\infty}\left|\frac{(2x)^{n+1}}{(2x)^n}\right| = |2x| < 1$$

when $|x| < \frac{1}{2}$

$\Rightarrow -\frac{1}{2} < x < \frac{1}{2}$

So the the series converges for x in $\underline{(-\frac{1}{2}, \frac{1}{2})}$.

The $\underline{\text{R.O.C. is } 1/2}$ (Center is at 0!).

b) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x+1)^n}{4^n} \Rightarrow$ ratio test centered at -1 $\lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{(x+1)^n} \right| = \left|\frac{x+1}{4}\right|$

So $\left|\frac{x+1}{4}\right| < 1$ when $|x+1| < 4 \Rightarrow$ $\boxed{x \text{ is in } (-5, 3).}$

The center is at $-1 \Rightarrow$ $\boxed{\text{R.O.C. is } 4.}$

c) $\sum_{n=1}^{\infty} \frac{3x^n}{(n+5)(n+10)} \Rightarrow$ ratio test, series is centered at 0.

$$\Rightarrow \lim_{n\to\infty} \left| \frac{\cancel{3}x^{n+1}}{(n+6)(n+11)} \cdot \frac{(n+5)(n+10)}{\cancel{3x^n}} \right| = \lim_{n\to\infty} \left| \frac{x \cdot (n+5)(n+10)}{(n+6)(n+11)} \right|$$

$= |x|$, $|x| < 1 \Rightarrow$ $\boxed{\begin{array}{l}\text{Interval of convergence is } (-1, 1).\\ \text{R.O.C. is } R=1\end{array}}$

d) $\sum_{n=0}^{\infty} \frac{(x+1)^n}{(n+1)!} \Rightarrow \lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{(n+2)!} \cdot \frac{\cancel{(n+1)!}}{\cancel{(x+1)^n}} \right| = \lim_{n\to\infty} \left| \frac{x+1}{n+2} \right| \to 0.$

$\boxed{\text{So this will converge for all } x!}$

5) a) $f(x) = e^x$. at $c=0$ $=1$

$f'(x) = e^x$ at $c=0$ $=1$

$f''(x) = e^x$ at $c=0$ $=1$.

At this point, the pattern is clear – $f^{(n)}(x) = e^x$, at $c=0 \Rightarrow =1$.

So $a_n = \frac{f^{(n)}(c)}{n!} = \frac{1}{n!} \Rightarrow \sum_{n=0}^{\infty} a_n (x-0)^n = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$

b) Now $f(x) = e^{-x^3} \Rightarrow$ Let $u = -x^3$.

$\Rightarrow e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} = \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n (x)^{3n}}{n!}}$

c) $f(x) = \frac{1}{x+1}$, center at 0. $\Rightarrow$ $f(0) = 1$.

$f'(x) = \frac{-1}{(x+1)^2}$, $f'(0) = \frac{-1}{1^2}$

(☆) $f''(x) = \frac{(-1)(-2)}{(x+1)^3}$, $f''(0) = \frac{(-1)(-2)}{1^3} = \frac{1 \cdot 2}{1}$

$f'''(x) = \frac{(-1)(-2)(-3)}{(x+1)^4}$, $f'''(0) = \frac{(-1)(-2)(-3)}{1^4} = \frac{-1\,(1)(2)(3)}{1}$

$f^{(4)}(x) = \frac{(-1)(-2)(-3)(-4)}{(x+1)^5}$, $f^{(4)}(0) = \frac{1 \cdot 2 \cdot 3 \cdot 4}{(1)^5} = \frac{1 \cdot 2 \cdot 3 \cdot 4}{1}$

$f^{(n)}(x) = (-1)^n \frac{n!}{1}$ -1 on the odd n's.

$\Rightarrow a_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^n n!}{n!} = (-1)^n$.

$\Rightarrow \sum_{n=0}^{\infty} a_n (x-0)^n = \boxed{\sum_{n=0}^{\infty} (-1)^n x^n.}$

d) Note that taking the derivative of $\frac{1}{(x+1)}$ will get us to $\frac{1}{(x+1)^3}$! (look at (☆) above)

$\Rightarrow$ if $f(x) = \frac{1}{x+1}$, $f'(x) = \frac{-1}{(x+1)^2}$, $f''(x) = \frac{2}{(x+1)^3}$.

$\Rightarrow$ Need to take derivative twice of $\frac{1}{x+1} = \sum_{n=0}^{\infty} (-1)^n (x)^n$, and divide by $2!$

1st derivative $\Rightarrow \left(\sum_{n=0}^{\infty} (-1)^n x^n\right)' = \sum_{n=1}^{\infty} (-1)^n \cdot n\, x^{n-1}$ note we start at $n=1$ now because the $n=0$ goes away in the derivative.

2nd derivative $\Rightarrow \left(\sum_{n=0}^{\infty} (-1)^n \cdot n\, x^{n-1}\right)' = \sum_{n=2}^{\infty} (-1)^n (n)(n-1) x^{n-2}$

$\Rightarrow \frac{1}{(x+1)^3} = \boxed{\frac{1}{2} \cdot \sum_{n=2}^{\infty} (-1)^n (n)(n-1)\, x^{n-2}}$