Key to Study Guide

1) a) $\vec{F} = (y+z)\vec{i} + (x+z)\vec{j} + (x+z)\vec{k}$

$\frac{\partial P}{\partial y} = 0 \neq \frac{\partial N}{\partial z} = 1 \Rightarrow$ not conservative.

b) $\vec{F} = e^{y+5z}(\vec{i} + x\vec{j} + 5x\vec{k})$

$\frac{\partial M}{\partial y} = e^{y+5z} = \frac{\partial N}{\partial x}$. ✓ $\quad \frac{\partial M}{\partial z} = 5e^{y+5z} = \frac{\partial P}{\partial x}$. ✓

$\frac{\partial N}{\partial z} = 5xe^{y+5z} = \frac{\partial P}{\partial y}$. ✓ $\Rightarrow$ conservative.

So to find f, $M = e^{y+5z} = \frac{\partial f}{\partial x} \Rightarrow \int M\,dx = xe^{y+5z} + C(y,z)$

$\Rightarrow \frac{\partial f}{\partial y} = xe^{y+5z} + \frac{\partial(C(y,z))}{\partial y} = N = xe^{y+5z}$

$\Rightarrow \frac{\partial(C(y,z))}{\partial y} = 0 \Rightarrow C(y,z) = C(z).$

$\Rightarrow \frac{\partial f}{\partial z} = 5xe^{y+5z} + \frac{\partial(C(z))}{\partial z} = P = 5xe^{y+5z} \Rightarrow \frac{\partial(C(z))}{\partial z} = 0.$

$\Rightarrow \boxed{f(x,y,z) = xe^{y+5z} + C}$

2) $\int_{(1,2,1)}^{(2,1,1)} (2x\ln y - yz)\,dx + \left(\frac{x^2}{y} - xz\right)dy - xy\,dz$

Check if conservative for shortcut $\Rightarrow \frac{\partial M}{\partial y} = \frac{2x}{y} - z = \frac{\partial N}{\partial x}$. ✓

$\frac{\partial M}{\partial z} = -y = \frac{\partial P}{\partial x}$. ✓ $\quad \frac{\partial N}{\partial z} = -x = \frac{\partial P}{\partial y}$. ✓ $\quad \Rightarrow$ conservative.

now find f — $M = 2x\ln y - yz = \frac{\partial f}{\partial x}$

$\Rightarrow \int M\,dx = x^2\ln y - xyz + C(y,z)$

$\frac{\partial f}{\partial y} = \frac{x^2}{y} - xz + \frac{\partial(C(y,z))}{\partial y} = N = \frac{x^2}{y} - xz \Rightarrow \frac{\partial(C(y,z))}{\partial y} = 0$

$\Rightarrow C(y,z) = C(z)$

$\frac{\partial f}{\partial z} = 0 - xy + \frac{\partial(C(z))}{\partial z} = P = -xy \Rightarrow \frac{\partial(C(z))}{\partial z} = 0 \Rightarrow C(z) = C$

$\Rightarrow f(x,y,z) = x^2\ln y - xyz + C.$

So $\displaystyle\int_{(1,2,1)}^{(2,1,1)} \nabla f = f(2,1,1) - f(1,2,1) = 4\cancel{\ln}\,1 - 2 - \ln 2 + 2$

$= -\ln 2.$

Since $\vec{F}$ is conservative, it is path independent, hence the value of the second path - or any other path, with the same starting and ending points is the same - $= -\ln 2$.

3) Circulation - use Green's - $\displaystyle\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^1\int_0^x (-2x-1)\,dy\,dx$ 

$= \boxed{-\frac{7}{6}}$

Flux - Green's - $\displaystyle\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA = \iint_R (1-2y)\,dA = \int_0^1\int_0^x 1-2y\,dy\,dx = \boxed{\frac{1}{6}}$

4) $\displaystyle\oint_C y^2\,dx + x^2\,dy$  use Green's again - note the form of the integral here, $\displaystyle\oint_C M\,dx + N\,dy$

$\Rightarrow \quad \displaystyle = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^1\int_0^{1-x} (2x - 2y) = \boxed{0}.$

5) the equation of the surface is $g(x,y,z) = x^2+y^2-z=0$.

$\Rightarrow \nabla g = 2x\vec{i} + 2y\vec{j} - 1\vec{k}. \quad |\nabla g| = \sqrt{4x^2+4y^2+1}$

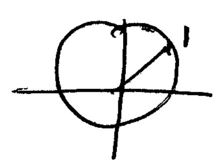
$\vec{n} = \dfrac{\nabla g}{|\nabla g|} = \dfrac{2x\vec{i}+2y\vec{j}-1\vec{k}}{\sqrt{4x^2+4y^2+1}}$

$d\sigma = \dfrac{|\nabla g|}{|\nabla g \cdot \vec{p}|}$ like usual - take $\vec{p} = \vec{k}$. $\Rightarrow \quad |\nabla g \cdot \vec{p}| = 1$

$d\sigma = \sqrt{4x^2+4y^2+1}\,dA$

$\Rightarrow$ Flux $= \displaystyle\iint_R \vec{F}\cdot\vec{n}\,d\sigma = \iint_R (8x^2+8y^2 - 2)\,dA$ ($|\nabla g|$ cancels).

$= \displaystyle\int_0^{2\pi}\int_0^1 (8r^2-2)\,r\,dr\,d\theta = 2\pi$ since $x^2+y^2=z$, $z=1$ - ~~take~~



~~$0\le\theta\le 2\pi$ $x = r\cos\theta$ $y = r\sin\theta$~~ ...

6) take the parametrization to be $x = x$
$y = y$
$z = 4 - y^2$

$\Rightarrow \quad \vec{r}(x,y) = x\vec{i} + y\vec{j} + (4-y^2)\vec{k}$

$0 \le x \le 1$
$-2 \le y \le 2.$

$\vec{r}_x = \vec{i}$ $\quad \vec{r}_y = \vec{j} - 2y\vec{k}$, $\quad \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & -2y \end{vmatrix} = 2y\vec{j} + \vec{k}$.

So surface area - $\iint_S |\vec{r}_x \times \vec{r}_y| \, d\sigma = \int_0^1 \int_{-2}^{2} \sqrt{4y^2+1} \, dy \, dx$

$= \frac{1}{2} \int_0^1 \frac{2y}{2} \sqrt{4y^2+1} + \frac{1}{2} \ln(2y + \sqrt{4y^2+1}) \Big|_{-2}^{2} dx$ (formula 21 in the book).

$= \frac{1}{2}(\ln(4+\sqrt{17}) - \ln(\sqrt{17}-4))$. (no need to worry about this kind of integral)

$\Rightarrow$ Flux - $F \cdot \frac{(\vec{r}_x \times \vec{r}_y)}{|\vec{r}_x \times \vec{r}_y|} \cdot |\vec{r}_x \times \vec{r}_y| \cdot dy\,dx = (2xy - 3z)\,dy\,dx = \iint_S \vec{F} \cdot \vec{n} \, d\sigma$

$= \int_0^1 \int_{-2}^{2} (2xy + 3y^2 - 12)\,dy\,dx = \boxed{-32}$.

7) curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & x \end{vmatrix} = \vec{i}(0-x) - \vec{j}(1-0) + \vec{k}(z-1)$

$\vec{n} = \frac{\nabla g}{|\nabla g|}$ (g in this case is the plane $x+y+z=1$).

$\vec{n} = \frac{1\vec{i} + 1\vec{j} + 1\vec{k}}{\sqrt{3}}$ $\quad \Rightarrow d\sigma = \frac{|\nabla g|}{|\nabla g \cdot \vec{k}|} = \frac{\sqrt{3}}{1}$.

$\Rightarrow \oint_C \vec{F} \cdot d\vec{r}$ (Stokes) $= \iint_R \nabla \times \vec{F} \cdot \vec{n} \, d\sigma = \int_0^1 \int_0^{1-x} (-x - 1 + z - 1)\,dy\,dx$

$= \int_0^1 \int_0^{1-x} (-x - 2 + (1-x-y))\,dy\,dx = \boxed{-1}$

8) $\nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = 15x^2 + 12y^2 + 3y^2 + \cancel{e^y \sin z} + 15z^2 - \cancel{e^y \sin z}$

$= 15x^2 + 15y^2 + 15z^2 = 15\rho^2$ (don't specify ρ - in between the spheres)

$\Rightarrow$ Flux $= \iiint_D 15\rho^2 \, dV = \int_0^{2\pi} \int_0^{\pi} \int_1^{\sqrt{2}} (15\rho^2)\, \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

$= \boxed{(48\sqrt{2} - 12)\pi}$