



## Chapter 4

# Prandtl's Boundary Layer Theory

Prandtl introduced boundary layer theory in 1905 to understand the flow of a slightly viscous fluid near a solid boundary. Prandtl's boundary layer theory is the original, and fundamental, example of a singular perturbation problem that can be treated by the method of matched asymptotic expansions.

### 4.1 The Navier-Stokes and Euler equations

The flow of an incompressible, viscous fluid is described by the incompressible Navier-Stokes equations,

$$\begin{aligned}\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p &= \varepsilon \Delta \mathbf{u}, \\ \nabla \cdot \mathbf{u} &= 0.\end{aligned}$$

Here,  $\mathbf{u}(\mathbf{x}, t)$  is the fluid velocity,  $p(\mathbf{x}, t)$  is the pressure, and  $\varepsilon = 1/\text{Re}$  is the dimensionless viscosity, or inverse Reynold's number. Setting  $\varepsilon = 0$ , we get the incompressible Euler equations,

$$\begin{aligned}\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p &= 0, \\ \nabla \cdot \mathbf{u} &= 0.\end{aligned}$$

Both the Navier-Stokes and the Euler equations are first order in time, and they require the same initial condition for the velocity,

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x).$$

No initial condition is required for the pressure.

The Navier-Stokes equations are a singular perturbation of the Euler equations because they contain higher-order spatial derivatives. As a result, the Navier-Stokes equations require different boundary conditions from the Euler equations to be well-posed.

Suppose that the fluid flows in a region  $\Omega$  with a stationary, solid boundary  $\partial\Omega$ . The appropriate boundary condition for the Navier-Stokes equations is the 'no-slip'

condition

$$\mathbf{u}(\mathbf{x}, t) = 0 \quad \text{for } \mathbf{x} \in \partial\Omega,$$

which means that a viscous fluid ‘sticks’ to the boundary. The appropriate boundary condition for the Euler equations is the ‘no-flow’ conditions,

$$\mathbf{u}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in \partial\Omega,$$

where  $\mathbf{n}$  is the unit normal vector to the boundary. Thus, for a viscous fluid both the normal and tangential velocities are zero at the boundary, whereas for an inviscid fluid only the normal velocity is zero.

Prandtl’s idea was that when the viscosity is small a thin boundary layer forms near the boundary in which the tangential velocity component adjusts rapidly from a nonzero value away from the boundary to a zero value on the boundary. Outside the boundary layer, the inviscid equations hold, and inside the boundary layer, viscosity becomes important because the solution varies rapidly in the direction normal to the boundary and the higher-order viscous terms  $\varepsilon\Delta\mathbf{u}$  become significant.

## 4.2 Two-dimensional boundary layer equations

We will consider two-dimensional flow over a flat boundary.\* We write the spatial coordinates as  $\mathbf{x} = (x, y)$  and the velocity  $\mathbf{u} = (u, v)$ , where  $u(x, y, t)$  and  $v(x, y, t)$  are the  $x$  and  $y$  velocity components, respectively. We suppose that there is a solid boundary is located at  $y = 0$ . The component form of the two-dimensional Navier-Stokes equations in  $y > 0$  is

$$\begin{aligned} u_t + uu_x + vv_y + p_x &= \varepsilon(u_{xx} + u_{yy}), \\ v_t + uv_x + vv_y + p_y &= \varepsilon(v_{xx} + v_{yy}), \\ u_x + v_y &= 0, \end{aligned}$$

with the ‘no-slip’ boundary conditions

$$u(x, 0, t) = 0, \quad v(x, 0, t) = 0,$$

and initial conditions

$$u(x, y, 0) = u_0(x, y), \quad v(x, y, 0) = v_0(x, y).$$

We assume that  $v_0(x, 0) = 0$ , so that the initial data is compatible with the ‘no-flow’ condition, but we do not necessarily assume that  $u_0(x, 0) = 0$ . For example, we may consider a plate that is set impulsively into motion at  $t = 0$ .

\*Since the boundary layer is thin, one obtains the same leading order boundary-layer equations for flow over a smooth curved boundary.

The leading-order outer solution in  $y > 0$ , which we still denote by  $(u, v, p)$ , satisfies the inviscid Euler equations obtained by setting  $\varepsilon = 0$ ,

$$\begin{aligned} u_t + uu_x + vv_y + p_x &= 0, \\ v_t + uv_x + vv_y + p_y &= 0, \\ u_x + v_y &= 0, \end{aligned}$$

with initial conditions

$$u(x, y, 0) = u_0(x, y), \quad v(x, y, 0) = v_0(x, y),$$

and the ‘no-flow’ boundary condition

$$v(x, 0, t) = 0.$$

These equations determine the outer solution. We cannot impose the full ‘no-slip’ condition, since the resulting inviscid problem is over-determined and in general  $u(x, 0, t) \neq 0$ .

Inside the boundary layer, we expect that: (a) the solution varies rapidly in  $y$ , since  $u$  has to adjust from a nonzero value in the outer, inviscid solution to a zero value on the boundary; (b) the transverse component  $v$  of the velocity is small, since it vanishes on the boundary and in the leading-order outer solution at  $y = 0$ ; (c) the solution varies on an order one longitudinal length-scale in  $x$ , and the longitudinal component  $u$  of the velocity is of the order one. (Here, we have implicitly non-dimensionalized lengths and velocities by characteristic values of the outer, inviscid solution.)

We therefore look for an inner solution of the form

$$\begin{aligned} u &= U(X, Y, T), \quad v = \eta V(X, Y, T), \quad p = P(X, Y, T), \\ X &= x, \quad Y = \frac{y}{\delta}, \quad T = t, \end{aligned}$$

where inner variables are denoted by capital letters, and  $\eta(\varepsilon)$  and  $\delta(\varepsilon)$  are small scaling parameters for the transverse velocity in the boundary layer and the boundary layer thickness, respectively. Using these expressions in the Navier-Stokes equations, we get

$$\begin{aligned} U_T + UU_X + \frac{\eta}{\delta} VU_Y + P_X &= \varepsilon U_{XX} + \frac{\varepsilon}{\delta^2} U_{YY}, \\ \eta V_T + \eta UV_X + \frac{\eta^2}{\delta} VV_Y + \frac{1}{\eta} P_Y &= \varepsilon \eta V_{XX} + \frac{\varepsilon \eta}{\delta^2} V_{YY}, \\ U_X + \frac{\eta}{\delta} V_Y &= 0, \end{aligned}$$

An examination of these equations shows that we have a dominant balance when

$$\eta = \delta, \quad \delta = \sqrt{\varepsilon}.$$

Thus, the boundary layer thickness is of the order  $\varepsilon^{1/2}L$ , where  $L$  is a length-scale characteristic of the external flow.

The scaled equations in the boundary layer are then

$$\begin{aligned} U_T + UU_X + VU_Y + P_X &= \varepsilon U_{XX} + U_{YY}, \\ \varepsilon (V_T + UV_X + VV_Y) + P_Y &= \varepsilon^2 V_{XX} + \varepsilon V_{YY}, \\ U_X + V_Y &= 0, \end{aligned}$$

The leading order inner equations are therefore

$$\begin{aligned} U_T + UU_X + VU_Y + P_X &= U_{YY}, \\ P_Y &= 0, \\ U_X + V_Y &= 0, \end{aligned}$$

These equations are the Prandtl boundary layer equations. The second equation implies that the pressure is independent of  $Y$ .

The inner boundary layer solution must match as  $Y \rightarrow \infty$  with the outer inviscid solution as  $y \rightarrow 0$ . It follows that

$$U(X, Y, T) \rightarrow U^*(X, T), \quad P(X, T) = P^*(X, T),$$

where  $U^*$ ,  $P^*$  are given in terms of the outer solution by

$$U^*(X, T) = u(X, 0, T), \quad P^*(X, T) = p(X, 0, T).$$

These functions satisfy the compatibility relation obtained by setting  $y = 0$  and  $v = 0$  in the inviscid equations

$$U_T^* + U^*U_X^* + P_X^* = 0. \quad (4.1)$$

It follows from the equation  $U_X + V_Y = 0$  that

$$V(X, Y, T) \sim U_X^*(X, T)Y \quad \text{as } Y \rightarrow \infty$$

which matches with the inner expansion as  $y \rightarrow 0$ ,

$$v(x, y, t) \sim v_y(x, 0, t)y = u_x(x, 0, t)y.$$

Thus, we do not require matching conditions on  $V$ .

The corresponding initial condition for  $U$  is

$$U(X, Y, 0) = u_0(X, 0).$$

More generally, if the original initial data contains a boundary layer, so that

$$u(x, y, 0) = u_0\left(x, \frac{y}{\varepsilon^{1/2}}\right),$$

then

$$U(X, Y, 0) = u_0(X, Y).$$

We do not require an initial condition for  $V$ .

Thus, the final unsteady boundary layer equations are

$$\begin{aligned} U_T + UU_X + VU_Y + P_X^* &= U_{YY}, \\ U_X + V_Y &= 0, \\ U(X, Y, T) &\rightarrow U^*(X, T), \quad \text{as } Y \rightarrow \infty, \\ U(X, Y, 0) &= u_0(X, Y), \end{aligned}$$

where  $u_0(X, Y)$  and  $U^*(X, T)$  are prescribed functions, and  $P^*(X, T)$  is given by (4.1).

For steady boundary layer flows, with  $U > 0$ , these equations reduce to

$$\begin{aligned} UU_X + VU_Y + P_X^* &= U_{YY}, \\ U_X + V_Y &= 0, \\ U(X, Y) &\rightarrow U^*(X) \quad \text{as } Y \rightarrow \infty, \\ U(X, Y) &\sim U_0(Y) \quad \text{as } X \rightarrow -\infty \end{aligned}$$

where  $P_X^* = -U^*U_X^*$ .

### 4.3 Limitations of Prandtl's boundary layer theory

The notion of a boundary layer is a crucial concept for understanding high-Reynolds number ( $\text{Re} \gg 1$ ) fluid flows. Nevertheless, the phenomena of turbulence and boundary layer separation lead to severe limitations in the use of boundary layer theory as a quantitative asymptotic theory.

Boundary layer theory applies in the limit  $\text{Re} \rightarrow \infty$ . In that limit however, all flows with non-zero vorticity (which includes boundary layer flows) are turbulent. Turbulent flows possess a complex and apparently random fine structure, whose length-scales depend on the Reynold's number. The simple scaling assumed above for laminar boundary layer flows therefore no longer applies. In typical problems, the boundary layer may become turbulent at Reynold's numbers of the order  $10^6$ , and a systematic mathematical theory for turbulent flows is still lacking.

Boundary layer separation is a phenomenon in which the boundary layer leaves the boundary and enters the interior of the fluid. When this happens, there is a strong coupling between the outer flow and the boundary layer, often accompanied by turbulence. Separation is reflected in the boundary layer equations by the formation of singularities, which typically occur unless  $U_Y > 0$  throughout the solution. For example, boundary layer separation does not occur in flow past a sufficiently streamlined body (such as an airfoil); in that case inviscid flow theory together with boundary layer theory provides a good quantitative approximation of the flow field and the viscous drag. On the other hand, boundary layer separation does occur for

flow past bluff bodies (such as a sphere), leading to problems which are still not resolved.

The case of 'weak' separation can be studied by the use of a more sophisticated asymptotic expansion (called 'triple-deck' theory because one introduces three different regions). In the case of incompressible fluids, this expansion leads to Prandtl equations in which the pressure  $P^*$  is no-longer specified *a priori*, but is related in a nonlocal way to the asymptotic behavior of  $U$  as  $Y \rightarrow \infty$ .