Chapter 3

Asymptotic Expansion of Integrals

In this chapter, we give some examples of asymptotic expansions of integrals. We do not attempt to give a complete discussion of this subject (see [4], [21] for more information).

3.1 Euler's integral

Consider the following integral (Euler, 1754):

$$I(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\, dt, \tag{3.1}$$

where $x \geq 0$.

First, we proceed formally. We use the power series expansion

$$\frac{1}{1+xt} = 1 - xt + x^2t^2 + \ldots + (-1)^n x^n t^n + \ldots \tag{3.2}$$

inside the integral in (3.1), and integrate the result term-by-term. Using the integral

$$\int_0^\infty t^n e^{-t}\, dx = n!,$$

we get

$$I(x) \sim 1 - x + 2!x^2 + \ldots + (-1)^n n! x^n + \ldots. \tag{3.3}$$

The coefficients in this power series grow factorially, and the terms diverge as $n \to \infty$. Thus, the series does not converge for any $x \neq 0$. On the other hand, the following proposition shows that the series is an asymptotic expansion of $I(x)$ as $x \to 0^+$, and the the error between a partial sum and the integral is less than the first term neglected in the asymptotic series. The proof also illustrates the use of integration by parts in deriving an asymptotic expansion.

Proposition 3.1 For $x \geq 0$ and $N = 0, 1, 2, \ldots$, we have

$$\left|I(x) - \left\{1 - x + \ldots + (-1)^N N! x^N\right\}\right| \leq (N+1)!\, x^{N+1}.$$

Proof. Integrating by parts in (3.1), we have

$$I(x) = 1 - x \int_0^\infty \frac{e^{-t}}{(1+xt)^2}\, dt.$$

After $N+1$ integrations by parts, we find that

$$I(x) = 1 - x + \ldots + (-1)^N N! x^N + R_{N+1}(x),$$

where

$$R_{N+1}(x) = (-1)^{N+1}(N+1)! x^{N+1} \int_0^\infty \frac{e^{-t}}{(1+xt)^{N+2}}\, dt.$$

Estimating R_{N+1} for $x \geq 0$, we find that

$$\begin{aligned} |R_{N+1}(x)| &\leq (N+1)! x^{N+1} \int_0^\infty e^{-t}\, dt \\ &\leq (N+1)!\, x^{N+1} \end{aligned}$$

which proves the result. Equation (3.1) shows that the partial sums oscillate above (N even) and below (N odd) the value of the integral. □

Heuristically, the lack of convergence of the series in (3.3) is a consequence of the fact that the power series expansion (3.2) does not converge over the whole integration region, but only when $0 \leq t < 1/x$. On the other hand, when x is small, the expansion is convergent over most of the integration region, and the integrand is exponentially small when it is not. This explains the accuracy of the resulting partial sum approximations.

The integral in (3.1) is not well-defined when $x < 0$ since then the integrand has a nonintegrable singularity at $t = -1/x$. The fact that $x = 0$ is a 'transition point' is associated with the lack of convergence of the asymptotic power series, because any convergent power series would converge in a disk (in $\mathbb{C}$) centered at $x = 0$.

Since the asymptotic series is not convergent, its partial sums do not provide an arbitrarily accurate approximation of $I(x)$ for a fixed $x > 0$. It is interesting to ask, however, what partial sum gives the the best approximation.

This occurs when n minimizes the remainder $R_{n+1}(x)$. The remainder decreases when $n \leq x$ and increases when $n + 1 > x$, so the best approximation occurs when $n + 1 \approx [1/x]$, and then $R_{n+1}(x) \approx (1/x)! x^{1/x}$. Using Stirling's formula (see Example 3.10),

$$n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n} \qquad \text{as } n \to \infty,$$

we find that the optimal truncation at $n \approx [1/x] - 1$ gives an error

$$R_n(x) \sim \sqrt{\frac{2\pi}{x}}\, e^{-1/x} \qquad \text{as } x \to 0^+.$$

Thus, even though each partial sum with a fixed number of terms is polynomially accurate in x, the optimal partial sum approximation is exponentially accurate.

Example 3.2 The partial sums

$$S_N(x) = 1 - x + \ldots + (-1)^N N! x^N$$

for $x = 0.1$ and $2 \leq N \leq 15$ are given in the following table (to an appropriate accuracy).

N	$S_N(0.1)$
2	0.9
3	0.92
4	0.914
5	0.9164
6	0.9152
7	0.91529
8	0.915416
9	0.915819
10	0.915463
11	0.915819
12	0.91542
13	0.9159
14	0.9153
15	0.9162

It follows that

$$0.91546 \leq I(0.1) \leq 0.91582.$$

Numerical integration shows that, to four significant figures,

$$I(0.1) \approx 0.9156.$$

In some problems, the exponentially small corrections to a power series asymptotic expansion contain important information. For example, the vanishing or nonvanishing of these corrections may correspond to the existence or nonexistence of particular types of solutions of PDEs, such as traveling waves or breathers. There exist methods of exponential asymptotics, or asymptotics beyond all orders, that can be used to compute exponentially small terms.

3.2 Perturbed Gaussian integrals

Consider the following integral

$$I(a,\varepsilon) = \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}ax^2 - \varepsilon x^4\right]\, dx, \tag{3.4}$$

where $a > 0$ and $\varepsilon \geq 0$. For $\varepsilon = 0$, this is a standard Gaussian integral, and

$$I(a,0) = \frac{1}{\sqrt{2\pi a}}.$$

For $\varepsilon > 0$, we cannot compute $I(a,\varepsilon)$ explicitly, but we can obtain an asymptotic expansion as $\varepsilon \to 0^+$.

First, we proceed formally. Taylor expanding the exponential with respect to ε,

$$\exp\left[-\frac{1}{2}ax^2 - \varepsilon x^4\right] = e^{-\frac{1}{2}ax^2}\left\{1 - \varepsilon x^4 + \frac{1}{2!}\varepsilon^2 x^8 + \ldots + \frac{(-1)^n}{n!}\varepsilon^n x^{4n} + \ldots\right\},$$

and integrating the result term-by-term, we get

$$I(a,\varepsilon) \sim \frac{1}{\sqrt{2\pi a}}\left\{1 - \varepsilon\langle x^4\rangle + \ldots + \frac{(-1)^n}{n!}\varepsilon^n\langle x^{4n}\rangle + \ldots\right\},, \tag{3.5}$$

where

$$\langle x^{4n}\rangle = \frac{\int_{-\infty}^{\infty} x^{4n} e^{-\frac{1}{2}ax^2}\, dx}{\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2}\, dx}.$$

We use a special case of Wick's theorem to calculate these integrals.

Proposition 3.3 For $m \in \mathbb{N}$, we have

$$\langle x^{2m}\rangle = \frac{(2m-1)!!}{a^m},$$

where

$$(2m-1)!! = 1 \cdot 3 \cdot 5 \ldots (2m-3) \cdot (2m-1).$$

Proof. Let

$$J(a,b) = \frac{\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2 + bx}\, dx}{\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2}\, dx}.$$

Differentiating $J(a,b)$ n-times with respect to b and setting $b = 0$, we find that

$$\langle x^n\rangle = \left.\frac{d^n}{db^n}J(a,b)\right|_{b=0}.$$

Writing

$$e^{-\frac{1}{2}ax^2 + bx} = e^{-\frac{1}{2}a(x-b)^2 + \frac{b^2}{2a}}$$

and making the change of variable $(x-b) \mapsto x$ in the numerator, we deduce that

$$J(a,b) = e^{\frac{b^2}{2a}}.$$

Hence,

$$\begin{aligned}\langle x^n \rangle &= \frac{d^n}{db^n}\left[e^{\frac{b^2}{2a}}\right]\Bigg|_{b=0} \\ &= \frac{d^n}{db^n}\left\{1 + \frac{b^2}{2a} + \ldots + \frac{1}{m!}\frac{b^{2m}}{(2a)^m} + \ldots\right\}\Bigg|_{b=0},\end{aligned}$$

which implies that

$$\langle x^{2m} \rangle = \frac{(2m)!}{(2a)^m m!}.$$

This expression is equivalent to the result. □

Using the result of this proposition in (3.5), we conclude that

$$I(a,\varepsilon) \sim \frac{1}{\sqrt{2\pi a}}\left[1 - \frac{3}{a^2}\varepsilon + \frac{105}{a^4}\varepsilon^2 + \ldots + a_n\varepsilon^n + \ldots\right] \qquad \text{as } \varepsilon \to 0^+, \tag{3.6}$$

where

$$a_n = \frac{(-1)^n(4n-1)!!}{n! a^{2n}}. \tag{3.7}$$

By the ratio test, the radius of convergence R of this series is

$$\begin{aligned}R &= \lim_{n\to\infty} \frac{(n+1)! a^{2n+2}(4n-1)!}{n! a^{2n}(4n+3)!!} \\ &= \lim_{n\to\infty} \frac{(n+1)a^2}{(4n+1)(4n+3)} \\ &= 0.\end{aligned}$$

Thus, the series diverges for every $\varepsilon > 0$, as could have been anticipated by the fact that $I(a,\varepsilon)$ is undefined for $\varepsilon < 0$.

The next proposition shows that the series is an asymptotic expansion of $I(a,\varepsilon)$ as $\varepsilon \to 0^+$.

Proposition 3.4 Suppose $I(a,\varepsilon)$ is defined by (3.4). For each $N = 0, 1, 2, \ldots$ and $\varepsilon > 0$, we have

$$\left| I(a,\varepsilon) - \sum_{n=0}^{N} a_n \varepsilon^n \right| \le C_{N+1}\varepsilon^{N+1}$$

where a_n is given by (3.7), and

$$C_{N+1} = \frac{1}{(N+1)!}\int_{-\infty}^{\infty} x^{4(N+1)} e^{-\frac{1}{2}ax^2}\, dx.$$

Proof. Taylor's theorem implies that for $y \geq 0$ and $N \in \mathbb{N}$

$$e^{-y} = 1 - y + \frac{1}{2!}y^2 + \ldots + \frac{(-1)^N}{N!}y^N + s_{N+1}(y),$$

where

$$s_{N+1}(y) = \frac{1}{(N+1)!}\, \frac{d^{N+1}}{dy^{N+1}}\left(e^{-y}\right)\Big|_{y=\eta} y^{N+1}$$

for some $0 \leq \eta \leq y$. Replacing y by εx^4 in this equation and estimating the remainder, we find that

$$e^{-\varepsilon x^4} = 1 - \varepsilon x^4 + \frac{1}{2!}\varepsilon^2 x^4 + \ldots + \frac{(-1)^N}{N!}\varepsilon^N x^{4N} + \varepsilon^{N+1} r_{N+1}(x), \tag{3.8}$$

where

$$|r_{N+1}(x)| \leq \frac{x^{4(N+1)}}{(N+1)!}.$$

Using (3.8) in (3.4), we get

$$I(a,\varepsilon) = \sum_{n=0}^{N} a_n \varepsilon^n + \varepsilon^{N+1} \int_{-\infty}^{\infty} r_{N+1}(x) e^{-\frac{1}{2}ax^2}\, dx.$$

It follows that

$$\begin{aligned} \left| I(a,\varepsilon) - \sum_{n=0}^{N} a_n \varepsilon^n \right| &\leq \varepsilon^{N+1} \int_{-\infty}^{\infty} |r_{N+1}(x)|\, e^{-\frac{1}{2}ax^2}\, dx \\ &\leq \varepsilon^{N+1} \frac{1}{(N+1)!} \int_{-\infty}^{\infty} x^{4(N+1)} e^{-\frac{1}{2}ax^2}\, dx, \end{aligned}$$

which proves the result. □

These expansions generalize to multi-dimensional Gaussian integrals, of the form

$$I(A,\varepsilon) = \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}x^T A x + \varepsilon V(x)\right) dx$$

where A is a symmetric $n \times n$ matrix, and to infinite-dimensional functional integrals, such as those given by the formal expression

$$I(\varepsilon) = \int \exp\left\{-\int \left(\frac{1}{2}|\nabla u(x)|^2 + \frac{1}{2}u^2(x) + \varepsilon V\left(u(x)\right)\right) dx\right\} Du$$

which appear in quantum field theory and statistical physics.

3.3 The method of stationary phase

The method of stationary phase provides an asymptotic expansion of integrals with a rapidly oscillating integrand. Because of cancelation, the behavior of such integrals is dominated by contributions from neighborhoods of the stationary phase points where the oscillations are the slowest.

Example 3.5 Consider the following Fresnel integral

$$I(\varepsilon) = \int_{-\infty}^{\infty} e^{it^2/\varepsilon}\, dt.$$

This oscillatory integral is not defined as an absolutely convergent integral, since $\left|e^{it^2/\varepsilon}\right| = 1$, but it can be defined as an improper integral,

$$I(\varepsilon) = \lim_{R\to\infty} \int_{-R}^{R} e^{it^2/\varepsilon}\, dt.$$

This convergence follows from an integration by parts:

$$\int_1^R e^{it^2/\varepsilon}\, dt = \left[\frac{\varepsilon}{2it} e^{it^2/\varepsilon}\right]_1^R + \int_1^R \frac{\varepsilon}{2it^2} e^{it^2/\varepsilon}\, dt.$$

The integrand oscillates rapidly away from the stationary phase point $t = 0$, and these parts contribute terms that are smaller than any power of ε, as we show below. The first oscillation near $t = 0$, where cancelation does not occur, has width of the order $\varepsilon^{1/2}$, so we expect that $I(\varepsilon) = O(\varepsilon^{1/2})$ as $\varepsilon \to 0$.

In fact, using contour integration and changing variables $t \mapsto e^{i\pi/4}s$ if $\varepsilon > 0$ and $t \mapsto E^{-i\pi/4}s$ if $\varepsilon < 0$, one can show that

$$\int_{-\infty}^{\infty} e^{it^2/\varepsilon}\, dt = \begin{cases} e^{i\pi/4}\sqrt{2\pi|\varepsilon|} & \text{if } \varepsilon > 0 \\ e^{-i\pi/4}\sqrt{2\pi|\varepsilon|} & \text{if } \varepsilon < 0 \end{cases}.$$

Next, we consider the integral

$$I(\varepsilon) = \int_{-\infty}^{\infty} f(t) e^{i\varphi(t)/\varepsilon}\, dt, \tag{3.9}$$

where $f : \mathbb{R} \to \mathbb{C}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ are smooth functions. A point $t = c$ is a stationary phase point if $\varphi'(c) = 0$. We call the stationary phase point nondegenerate if $\varphi''(c) \neq 0$.

Suppose that I has a single stationary phase point at $t = c$, which is nondegenerate. (If there are several such points, we simply add together the contributions from each one.) Then, using the idea that only the part of the integrand near the stationary phase point $t = c$ contributes significantly, we can Taylor expand the function f and the phase φ to approximate $I(\varepsilon)$ as follows:

$$I(\varepsilon) \quad \sim \quad \int f(c) \exp \frac{i}{\varepsilon}\left[\varphi(c) + \frac{1}{2}\varphi''(c)(t-c)^2\right] dt$$

$$\begin{aligned} &\sim f(c)e^{i\varphi(c)/\varepsilon}\int \exp\left[\frac{i\varphi''(c)}{2\varepsilon}s^2\right] ds \\ &\sim \sqrt{\frac{2\pi\varepsilon}{|\varphi''(c)|}}f(c)e^{i\varphi(c)/\varepsilon+i\sigma\pi/4}, \end{aligned}$$

where

$$\sigma = \operatorname{sgn}\varphi''(c).$$

More generally, we consider the asymptotic behavior as $\varepsilon \to 0$ of an integral of the form

$$I(x,\varepsilon) = \int A(x,\xi)e^{i\varphi(x,\xi)/\varepsilon}\, d\xi, \tag{3.10}$$

where $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^m$. We assume that

$$\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \qquad A : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{C}$$

are smooth (C^∞) functions, and that the support of A,

$$\operatorname{supp} A = \overline{\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^m \mid A(x,\xi) \neq 0\}},$$

is a compact subset of $\mathbb{R}^n \times \mathbb{R}^m$.

Definition 3.6 A stationary, or critical, point of the phase φ is a point $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^m$ such that

$$\frac{\partial\varphi}{\partial\xi}(x,\xi) = 0. \tag{3.11}$$

A stationary phase point is nondegenerate if

$$\frac{\partial^2\varphi}{\partial\xi^2} = \left(\frac{\partial^2\varphi}{\partial\xi_i\partial\xi_j}\right)_{i,j=1,\ldots,m}$$

is invertible at the stationary phase point.

Proposition 3.7 If the support of A contains no stationary points of φ, then

$$I(x,\varepsilon) = O\left(\varepsilon^n\right) \qquad \text{as } \varepsilon \to 0$$

for every $n \in \mathbb{N}$.

Proof. Rewriting the integral in (3.10), and integrating by parts, we have

$$\begin{aligned} I(x,\varepsilon) &= -i\varepsilon\int A\frac{\partial\varphi}{\partial\xi}\cdot\frac{\partial}{\partial\xi}\left[e^{i\varphi/\varepsilon}\right]\left|\frac{\partial\varphi}{\partial\xi}\right|^{-2} d\xi \\ &= i\varepsilon\int \frac{\partial}{\partial\xi}\cdot\left[A\left|\frac{\partial\varphi}{\partial\xi}\right|^{-2}\frac{\partial\varphi}{\partial\xi}\right]e^{i\varphi/\varepsilon}\, d\xi \\ &= O(\varepsilon). \end{aligned}$$

Applying this argument n times, we get the result. □

The implicit function theorem implies there is a unique local smooth solution of (3.11) for ξ in a neighborhood $U \times V \subset \mathbb{R}^n \times \mathbb{R}^m$ We write this stationary phase point as as $\xi = \overline{\xi}(x)$, where $\overline{\xi} : U \to V$. We may reduce the case of multiple nondegenerate critical points to this one by means of a partition of unity, and may also suppose that $\operatorname{supp} A \subset U \times V$. According to the Morse lemma, there is a local change of coordinates $\xi \mapsto \eta$ near a nondegenerate critical point such that

$$\varphi(x,\xi) = \varphi\left(x, \overline{\xi}(x)\right) + \frac{1}{2}\frac{\partial^2 \varphi}{\partial \xi^2}\left(x, \overline{\xi}(x)\right) \cdot (\eta, \eta).$$

Making this change of variables in (3.10), and evaluating the resulting Fresnel integral, we get the following stationary phase formula [10].

Theorem 3.8 Let $I(x,\varepsilon)$ be defined by (3.10), where φ is a smooth real-valued function with a nondegenerate stationary phase point at $(x, \overline{\xi}(x))$, and A is a compactly supported smooth function whose support is contained in a sufficiently small neighborhood of the stationary phase point. Then, as $\varepsilon \to 0$,

$$I(x,\varepsilon) \sim \frac{(2\pi\varepsilon)^{n/2}}{\sqrt{\det \left|\frac{\partial^2 \varphi}{\partial \xi^2}\right|_{\xi=\overline{\xi}(x)}}} e^{i\varphi(x,\overline{\xi}(x))/\varepsilon + i\pi\sigma/4} \sum_{p=0}^{\infty} (i\varepsilon)^p R_p(x),$$

where

$$\sigma = \operatorname{sgn}\left(\frac{\partial^2 \varphi}{\partial \xi^2}\right)_{\xi=\overline{\xi}(x)}$$

is the signature of the matrix (the difference between the number of positive and negative eigenvalues), $R_0 = 1$, and

$$R_p(x) = \sum_{|k| \le 2p} R_{pk}(x) \left.\frac{\partial^k A}{\partial \xi^k}\right|_{\xi=\overline{\xi}(x)},$$

where the R_{pk} are smooth functions depending only on φ.

3.4 Airy functions and degenerate stationary phase points

The behavior of the integral in (3.10) is more complicated when it has degenerate stationary phase points. Here, we consider the simplest case, where $\xi \in \mathbb{R}$ and two stationary phase points coalesce. The asymptotic behavior of the integral in a neighborhood of the degenerate critical point is then described by an Airy function.

Airy functions are solutions of the ODE

$$y'' = xy. \tag{3.12}$$

The behavior of these functions is oscillatory as $x \to -\infty$ and exponential as $x \to \infty$. They are the most basic functions that exhibit a transition from oscillatory to exponential behavior, and because of this they arise in many applications (for example, in describing waves at caustics or turning points).

Two linearly independent solutions of (3.12) are denoted by $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$. The solution $\mathrm{Ai}(x)$ is determined up to a constant normalizing factor by the condition that $\mathrm{Ai}(x) \to 0$ as $x \to \infty$. It is conventionally normalized so that

$$\mathrm{Ai}(0) = \frac{1}{3^{2/3}}\Gamma\left(\frac{2}{3}\right),$$

where Γ is the Gamma-function. This solution decays exponentially as $x \to \infty$ and oscillates, with algebraic decay, as $x \to -\infty$ [16],

$$\mathrm{Ai}(x) \sim \begin{cases} \frac{1}{2}\pi^{-1/2}x^{-1/4}\exp[-2x^{3/2}/3] & \text{as } x \to \infty, \\ \pi^{-1/2}(-x)^{-1/4}\sin[2(-x)^{3/2}/3 + \pi/4] & \text{as } x \to -\infty. \end{cases}$$

The solution $\mathrm{Bi}(x)$ grows exponentially as $x \to \infty$.

We can derive these results from an integral representation of $\mathrm{Ai}(x)$ that is obtained by taking the Fourier transform of (3.12).* Let $\widehat{y} = \mathcal{F}[y]$ denote the Fourier transform of y,

$$\begin{aligned} \widehat{y}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} y(x)e^{-ikx}\,dx, \\ y(x) &= \int_{-\infty}^{\infty} \widehat{y}(k)e^{ikx}\,dk. \end{aligned}$$

Then

$$\mathcal{F}[y''] = -k^2\widehat{y}, \quad \mathcal{F}[-ixy] = \widehat{y}'.$$

Fourier transforming (3.12), we find that

$$-k^2\widehat{y} = i\widehat{y}'.$$

Solving this first-order ODE, we get

$$\widehat{y}(k) = ce^{ik^3/3},$$

so y is given by the oscillatory integral

$$y(x) = c\int_{-\infty}^{\infty} e^{i\left(kx + k^3/3\right)}\,dk.$$

*We do not obtain Bi by this method because it grows exponentially as $x \to \infty$, which is too fast for its Fourier transform to be well-defined, even as a tempered distribution.

The standard normalization for the Airy function corresponds to $c = 1/(2\pi)$, and thus

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(kx+k^3/3\right)}\,dk. \tag{3.13}$$

This oscillatory integral is not absolutely convergent, but it can be interpreted as the inverse Fourier transform of a tempered distribution. The inverse transform is a C^∞ function that extends to an entire function of a complex variable, as can be seen by shifting the contour of integration upwards to obtain the absolutely convergent integral representation

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty+i\eta}^{\infty+i\eta} e^{i\left(kx+k^3/3\right)}\,dk.$$

Just as the Fresnel integral with a quadratic phase, provides an approximation near a nondegenerate stationary phase point, the Airy integral with a cubic phase provides an approximation near a degenerate stationary phase point in which the third derivative of the phase in nonzero. This occurs when two nondegenerate stationary phase points coalesce.

Let us consider the integral

$$I(x,\varepsilon) = \int_{-\infty}^{\infty} f(x,t)e^{i\varphi(x,t)/\varepsilon}\,dt.$$

Suppose that we have nondegenerate stationary phase points at

$$t = t_{\pm}(x)$$

for $x < x_0$, which are equal when $x = x_0$ so that $t_{\pm}(x_0) = t_0$. We assume that

$$\varphi_t\left(x_0,t_0\right) = 0, \quad \varphi_{tt}\left(x_0,t_0\right) = 0, \quad \varphi_{ttt}\left(x_0,t_0\right) \neq 0.$$

Then Chester, Friedman, and Ursell (1957) showed that in a neighborhood of (x_0, t_0) there is a local change of variables $t = \tau(x,s)$ and functions $\psi(x)$, $\rho(x)$ such that

$$\varphi(x,t) = \psi(x) + \rho(x)s + \frac{1}{3}s^3.$$

Here, we have $\tau(x_0, 0) = t_0$ and $\rho(x_0) = 0$. The stationary phase points correspond to $s = \pm\sqrt{-\rho(x)}$, where $\rho(x) < 0$ for $x < x_0$.

Since the asymptotic behavior of the integral as $\varepsilon \to 0$ is dominated by the contribution from the neighborhood of the stationary phase point, we expect that

$$\begin{aligned} I(x,\varepsilon) \quad &\sim \quad \int_{-\infty}^{\infty} f\left(x,\tau(x,s)\right)\tau_s(x,s)e^{i\left[\psi(x)+\rho(x)s+\frac{1}{3}s^3\right]/\varepsilon}\,ds \\ &\sim \quad f\left(x_0,t_0\right)\tau_s(x_0,0)e^{i\psi(x)/\varepsilon}\int_{-\infty}^{\infty} e^{i\left[\rho(x)s+\frac{1}{3}s^3\right]/\varepsilon}\,ds \end{aligned}$$

$$
\begin{aligned}
&\sim \quad \varepsilon^{1/3} f\left(x_0, t_0\right) \tau_s(x_0, 0) e^{i\psi(x)/\varepsilon} \int_{-\infty}^{\infty} e^{i\left[\varepsilon^{-2/3}\rho(x)k + \frac{1}{3}k^3\right]}\, dk \\
&\sim \quad 2\pi \varepsilon^{1/3} f\left(x_0, t_0\right) \tau_s(x_0, 0) e^{i\psi(x)/\varepsilon} \mathrm{Ai}\left(\frac{\rho(x)}{\varepsilon^{2/3}}\right),
\end{aligned}
$$

where we have made the change of variables $s = \varepsilon^{1/3}k$, and used the definition of the Airy function.

More generally, we have the following result. For the proof, see [10].

Theorem 3.9 Let $I(x, \varepsilon)$ be defined by (3.10), where $\varphi(x, \xi)$, with $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}$, is a smooth, real-valued function with a degenerate stationary phase point at $(x, \bar{\xi}(x))$. Suppose that

$$
\frac{\partial \varphi}{\partial \xi} = 0, \quad \frac{\partial^2 \varphi}{\partial \xi^2} = 0, \quad \frac{\partial^3 \varphi}{\partial \xi^3} \neq 0,
$$

at $\xi = \bar{\xi}(x)$, and $A(x, \xi)$ is a smooth function whose support is contained in a sufficiently small neighborhood of the degenerate stationary phase point. Then there are smooth real-valued functions $\psi(x)$, $\rho(x)$, and smooth functions $A_k(x)$, $B_k(x)$ such that

$$
I(x, \varepsilon) \sim \left[\varepsilon^{1/3} \mathrm{Ai}\left(\frac{\rho(x)}{\varepsilon^{2/3}}\right) \sum_{k=0}^{\infty} A_k(x) + i\varepsilon^{2/3} \mathrm{Ai}'\left(\frac{\rho(x)}{\varepsilon^{2/3}}\right) \sum_{k=0}^{\infty} B_k(x)\right] e^{i\psi(x)/\varepsilon}
$$

as $\varepsilon \to 0$.

3.4.1 *Dispersive wave propagation*

An important application of the method of stationary phase concerns the long-time, or large-distance, behavior of linear dispersive waves. Kelvin (1887) originally developed the method for this purpose, following earlier work by Cauchy, Stokes, and Riemann. He used it to study the pattern of dispersive water waves generated by a ship in steady motion, and showed that at large distances from the ship the waves form a wedge with a half-angle of $\sin^{-1}(1/3)$, or approximately $19.5°$.

As a basic example of a dispersive wave equation, we consider the following IVP (initial value problem) for the linearized KdV (Korteweg-de Vries), or Airy, equation,

$$
\begin{aligned}
u_t &= u_{xxx}, \\
u(x, 0) &= f(x).
\end{aligned}
$$

This equation provides an asymptotic description of linear, unidirectional, weakly dispersive long waves; for example, shallow water waves.

We assume for simplicity that the initial data $f : \mathbb{R} \to \mathbb{R}$ is a Schwarz function, meaning that it is smooth and decays, together with all its derivatives, faster than any polynomial as $|x| \to \infty$.

We use $\widehat{u}(k,t)$ to denote the Fourier transform of $u(x,t)$ with respect to x,

$$u(x,t) = \int_{-\infty}^{\infty} \widehat{u}(k,t)e^{ikx}\,dk,$$
$$\widehat{u}(k,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{-ikx}\,dx.$$

Then $\widehat{u}(k,t)$ satisfies

$$\widehat{u}_t + ik^3\widehat{u} = 0,$$
$$\widehat{u}(k,0) = \widehat{f}(k).$$

The solution of this equation is

$$\widehat{u}(k,t) = \widehat{f}(k)e^{-i\omega(k)t},$$

where

$$\omega(k) = k^3.$$

The function $\omega : \mathbb{R} \to \mathbb{R}$ gives the (angular) frequency $\omega(k)$ of a wave with wavenumber k, and is called the *dispersion relation* of the KdV equation.

Inverting the Fourier transform, we find that the solution is given by

$$u(x,t) = \int_{-\infty}^{\infty} \widehat{f}(k)e^{ikx-i\omega(k)t}\,dk.$$

Using the convolution theorem, we can write this solution as

$$u(x,t) = f * g(x,t),$$

where the star denotes convolution with respect to x, and

$$g(x,t) = \frac{1}{(3t)^{1/3}}\mathrm{Ai}\left(-\frac{x}{(3t)^{1/3}}\right)$$

is the Green's function of the Airy equation.

We consider the asymptotic behavior of this solution as $t \to \infty$ with $x/t = v$ fixed. This limit corresponds to the large-time limit in a reference frame moving with velocity v. Thus, we want to find the behavior as $t \to \infty$ of

$$u(vt,t) = \int_{-\infty}^{\infty} \widehat{f}(k)e^{i\varphi(k,v)t}\,dk, \tag{3.14}$$

where

$$\varphi(k,v) = kv - \omega(k).$$

The stationary phase points satisfy $\varphi_k = 0$, or

$$v = \omega'(k).$$

The solutions are the wavenumbers k whose group velocity $\omega'(k)$ is equal to v. It follows that

$$3k^2 = v.$$

If $v < 0$, then there are no stationary phase points, and $u(vt, t) = o(t^{-n})$ as $t \to \infty$ for any $n \in \mathbb{N}$.

If $v > 0$, then there are two nondegenerate stationary phase points at $k = \pm k_0(v)$, where

$$k_0(v) = \sqrt{\frac{v}{3}}.$$

These two points contribute complex conjugate terms, and the method of stationary phase implies that

$$u(vt, t) \sim \sqrt{\frac{2\pi}{|\omega''(k_0)|t}}\,\widehat{f}(k_0)e^{i\varphi(k_0,v)t - i\pi/4} + \text{c.c.} \qquad \text{as } t \to \infty.$$

The energy in the wave-packet therefore propagates at the group velocity $C = \omega'(k)$,

$$C = 3k^2,$$

rather than the phase velocity $c = \omega/k$,

$$c = k^2.$$

The solution decays at a rate of $t^{-1/2}$, in accordance with the linear growth in t of the length of the wavetrain and the conservation of energy,

$$\int_{-\infty}^{\infty} u^2(x, t)\, dt = \text{constant}.$$

The two stationary phase points coalesce when $v = 0$, and then there is a single degenerate stationary phase point. To find the asymptotic behavior of the solution when v is small, we make the change of variables

$$k = \frac{\xi}{(3t)^{1/3}}$$

in the Fourier integral solution (3.14), which gives

$$u(x, t) = \frac{1}{(3t)^{1/3}} \int_{-\infty}^{\infty} \widehat{f}\left(\frac{\xi}{(3t)^{1/3}}\right) e^{-i\left(\xi w + \frac{1}{3}\xi^3\right)}\, d\xi,$$

where

$$w = -\frac{t^{2/3}v}{3^{1/3}}.$$

It follows that as $t \to \infty$ with $t^{2/3}v$ fixed,

$$u(x,t) \sim \frac{2\pi}{(3t)^{1/3}}\widehat{f}(0)\operatorname{Ai}\left(-\frac{t^{2/3}v}{3^{1/3}}\right).$$

Thus the transition between oscillatory and exponential behavior is described by an Airy function. Since $v = x/t$, the width of the transition layer is of the order $t^{1/3}$ in x, and the solution in this region is of the order $t^{-1/3}$. Thus it decays more slowly and is larger than the solution elsewhere.

Whitham [20] gives a detailed discussion of linear and nonlinear dispersive wave propagation.

3.5 Laplace's Method

Consider an integral

$$I(\varepsilon) = \int_{-\infty}^{\infty} f(t)e^{\varphi(t)/\varepsilon}\,dt,$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{C}$ are smooth functions, and ε is a small positive parameter. This integral differs from the stationary phase integral in (3.9) because the argument of the exponential is real, not imaginary. Suppose that φ has a global maximum at $t = c$, and the maximum is nondegenerate, meaning that $\varphi''(c) < 0$. The dominant contribution to the integral comes from the neighborhood of $t = c$, since the integrand is exponentially smaller in ε away from that point. Taylor expanding the functions in the integrand about $t = c$, we expect that

$$\begin{aligned} I(\varepsilon) &\sim \int f(c)e^{\left[\varphi(c)+\frac{1}{2}\varphi''(c)(t-c)^2\right]/\varepsilon}\,dt \\ &\sim f(c)e^{\varphi(c)/\varepsilon}\int_{-\infty}^{\infty} e^{\frac{1}{2}\varphi''(c)(t-c)^2/\varepsilon}\,dt. \end{aligned}$$

Using the standard integral

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}at^2}\,dt = \sqrt{\frac{2\pi}{a}},$$

we get

$$I(\varepsilon) \sim f(c)\left(\frac{2\pi\varepsilon}{|\varphi''(c)|}\right)^{1/2} e^{\varphi(c)/\varepsilon} \quad \text{as } \varepsilon \to 0^+.$$

This result can proved under suitable assumptions on f and φ, but we will not give a detailed proof here (see [17], for example).

Example 3.10 The Gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ is defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt.$$

Integration by parts shows that if $n \in \mathbb{N}$, then

$$\Gamma(n+1) = n!.$$

Thus, the Gamma function extends the factorial function to arbitrary positive real numbers. In fact, the Gamma function can be continued to an analytic function

$$\Gamma : \mathbb{C} \setminus \{0, -1, -2, \ldots\} \to \mathbb{C}$$

with simple poles at $0, -1, -2, \ldots$.

Making the change of variables $t = xs$, we can write the integral representation of Γ as

$$\Gamma(x) = x^x \int_0^\infty \frac{1}{s} e^{x\varphi(s)} \, ds,$$

where

$$\varphi(s) = -s + \log s.$$

The phase $\varphi(s)$ has a nondegenerate maximum at $s = 1$, where $\varphi(1) = -1$, and $\varphi''(1) = -1$. Using Laplace's method, we find that

$$\Gamma(x) \sim \left(\frac{2\pi}{x}\right)^{1/2} x^x e^{-x} \quad \text{as } x \to \infty.$$

In particular, setting $x = n + 1$, and using the fact that

$$\lim_{n \to \infty} \left(\frac{n+1}{n}\right)^n = e,$$

we obtain Stirling's formula for the factorial,

$$n! \sim (2\pi)^{1/2} n^{n+1/2} e^{-n} \quad \text{as } n \to \infty.$$

This expansion of the Γ-function can be continued to higher orders to give:

$$\Gamma(x) \sim \left(\frac{2\pi}{x}\right)^{1/2} x^x e^{-x} \left[1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \ldots\right] \quad \text{as } x \to \infty,$$

$$a_1 = \frac{1}{12}, \quad a_2 = \frac{1}{288}, \quad a_3 = -\frac{139}{51,840}, \quad \ldots.$$

3.5.1 *Multiple integrals*

Proposition 3.11 Let A be a positive-definite $n \times n$ matrix. Then

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}x^T A x}\, dx = \frac{(2\pi)^{n/2}}{|\det A|^{1/2}}.$$

Proof. Since A is positive-definite (and hence symmetric) there is an orthogonal matrix S and a positive diagonal matrix $D = \operatorname{diag}[\lambda_1, \dots, \lambda_n]$ such that

$$A = S^T D S.$$

We make the change of variables $y = Sx$. Since S is orthogonal, we have $|\det S| = 1$, so the Jacobian of this transformation is 1. We find that

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-\frac{1}{2}x^T A x}\, dx &= \int_{\mathbb{R}^n} e^{-\frac{1}{2}y^T D y}\, dy \\ &= \prod_{i=1}^{n} \int_{\mathbb{R}} e^{-\frac{1}{2}\lambda_i y_i^2}\, dy_i \\ &= \frac{(2\pi)^{n/2}}{(\lambda_1 \dots \lambda_n)^{1/2}} \\ &= \frac{(2\pi)^{n/2}}{|\det A|^{1/2}}.\end{aligned}$$

□

Now consider the multiple integral

$$I(\varepsilon) = \int_{\mathbb{R}^n} f(t) e^{\varphi(t)/\varepsilon}\, dt.$$

Suppose that $\varphi : \mathbb{R}^n \to \mathbb{R}$ has a nondegenerate global maximum at $t = c$. Then

$$\varphi(t) = \varphi(c) + \frac{1}{2} D^2\varphi(c) \cdot (t - c, t - c) + O(|t - c|^3) \quad \text{as } t \to c.$$

Hence, we expect that

$$I(\varepsilon) \sim \int_{\mathbb{R}^n} f(c) e^{\left[\varphi(c) + \frac{1}{2}(t-c)^T A (t-c)\right]/\varepsilon}\, dt,$$

where A is the matrix of $D^2\varphi(c)$, with components

$$A_{ij} = \frac{\partial^2 \varphi}{\partial t_i \partial t_j}(c).$$

Using the previous proposition, we conclude that

$$I(\varepsilon) \sim \frac{(2\pi)^{n/2}}{\left|\det D^2\varphi(c)\right|^{1/2}} f(c) e^{\varphi(c)/\varepsilon}.$$

3.6 The method of steepest descents

Consider a contour integral of the form

$$I(\lambda) = \int_C f(z) e^{\lambda h(z)}\, dz,$$

where C is a contour in the complex plane, and $f, h : \mathbb{C} \to \mathbb{C}$ are analytic functions.

If $h(x+iy) = \varphi(x,y) + i\psi(x,y)$ is analytic, then $\varphi, \psi : \mathbb{R}^2 \to \mathbb{R}$ have no maxima or minima, and critical points where $h'(z) = 0$ are saddle points of φ, ψ. The curves $\varphi =$ constant, $\psi =$ constant are orthogonal except at critical points.

The idea of the method of steepest descents is to deform the contour C into a steepest descent contour passing through a saddle point on which φ has a maximum and $\psi =$ constant, so the contour is orthogonal to the level curves of φ. We then apply Laplace's method to the resulting integral. We will illustrate this idea by deriving the asymptotic behavior of the Airy function, given by (3.13)

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\left(kx + k^3/3\right)}\, dk.$$

To obtain the asymptotic behavior of $\mathrm{Ai}(x)$ as $x \to \infty$, we put this integral representation in a form that is suitable for the method of steepest descents. Setting $k = x^{1/2} z$, we find that

$$\mathrm{Ai}(x) = \frac{1}{2\pi} x^{1/2} I\left(x^{3/2}\right),$$

where

$$I(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda\left[z + \frac{1}{3}z^3\right]}\, dz.$$

The phase

$$h(z) = i\left(z + \frac{1}{3}z^3\right)$$

has critical points at $z = \pm i$.

Writing $h = \varphi + i\psi$ in terms of its real and imaginary parts, we have

$$\varphi(x,y) = -y\left(1 + x^2 - \frac{1}{3}y^2\right),$$
$$\psi(x,y) = x\left(1 + \frac{1}{3}x^2 - y^2\right).$$

The steepest descent contour $\psi(x,y) = 0$ through $z = i$, or $(x,y) = (0,1)$, is

$$y = \sqrt{1 + \frac{1}{3}x^2}.$$

When $\lambda > 0$, we can deform the integration contour $(-\infty, \infty)$ upwards to this steepest descent contour C, since the integrand decays exponentially as $|z| \to \infty$ in the upper-half plane. Thus,

$$I(\lambda) = \int_C e^{i\lambda\left[z+\frac{1}{3}z^3\right]}\, dz.$$

We parameterize C by $z(t) = x(t) + iy(t)$, where

$$x(t) = \sqrt{3}\sinh t, \quad y(t) = \cosh t.$$

Then we find that

$$I(\lambda) = \int_{-\infty}^{\infty} f(t) e^{i\lambda\varphi(t)}\, dt,$$

where

$$f(t) = \sqrt{3}\cosh t + i \sinh t,$$
$$\varphi(t) = \cosh t \left[2 - \frac{8}{3}\cosh^2 t\right].$$

The maximum of $\varphi(t)$ occurs at $t = 0$, where

$$\varphi(0) = -2/3, \quad \varphi'(0) = 0, \quad \varphi''(0) = -6.$$

Laplace's method implies that

$$\begin{aligned} I(\lambda) &\sim f(0)\left(\frac{2\pi}{-\lambda\varphi''(0)}\right)^{1/2} e^{\lambda\varphi(0)} \\ &\sim \left(\frac{\pi}{\lambda}\right)^{1/2} e^{-2\lambda/3}. \end{aligned}$$

It follows that

$$\mathrm{Ai}(x) \sim \frac{1}{2\pi^{1/2}x^{1/4}} e^{-2x^{3/2}/3} \qquad \text{as } x \to \infty. \tag{3.15}$$

Using the method of stationary phase, one can show from (3.13) that the asymptotic behavior of the Airy function as $x \to -\infty$ is given by

$$\mathrm{Ai}(x) \sim \frac{1}{\pi^{1/2}|x|^{1/4}} \sin\left[\frac{2}{3}|x|^{3/2} + \frac{\pi}{4}\right]. \tag{3.16}$$

This result is an example of a connection formula. It gives the asymptotic behavior as $x \to -\infty$ of the solution of the ODE (3.12) that decays exponentially as $x \to \infty$. This connection formula is derived using the integral representation (3.13), which provides global information about the solution.