Chapter 4

The Method of Matched Asymptotic Expansions: ODEs

Many singularly perturbed differential equations have solutions that change rapidly in a narrow region. This may occur in an initial layer where there is a rapid adjustment of initial conditions to a quasi-steady state, in a boundary layer where the solution away from the boundary adjusts to a boundary condition, or in an interior layer such as a propagating wave-front.

These problems can be analyzed using the method of matched asymptotic expansions (MMAE), in which we construct different asymptotic solutions inside and outside the region of rapid change, and 'match' them together to determine a global solution. A typical feature of this type of problem is a reduction in the order of the differential equation in the unperturbed problem, leading to a reduction in the number of initial or boundary conditions that can be imposed upon the solution. For additional information, see [3], [19].

4.1 Enzyme kinetics

Enzymes are proteins that act as catalysts. (There are also a smaller number of enzymes, called ribozymes, that contain catalytic RNA molecules.) A substance that is acted upon by an enzyme to create a product is called a substrate. Enzymes are typically very specific and highly efficient catalysts — tiny concentrations of enzymes compared with substrate concentrations are required.

For example, the enzyme catalase catalyzes the decomposition of hydrogen peroxide into water and oxygen, and one molecule of catalase can break up 40 million molecules of hydrogen peroxide each second. As another example, carbonic anhydrase occurs in red blood cells, where it catalyzes the reaction $CO_2 + H_2O \leftrightarrow H_2CO_3$ that enables the cells to transport carbon dioxide from the tissues to the lungs. One molecule of carbonic anhydrase can process one million molecules of CO_2 each second.

Michaelis and Menton (1913) proposed a simple model of enzymatically controlled reactions, in which the enzyme E and substrate S combine to form a complex C, and the complex breaks down irreversibly into the enzyme and a product P. Symbolically, we have

$$E + S \longleftrightarrow C \longrightarrow E + P.$$

We let

$$e = [E], \quad s = [S], \quad c = [C], \quad p = [P],$$

denote the concentrations of the corresponding species.

According to the law of mass action, the rate of a reaction is proportional to the product of the concentrations of the species involved, so that

$$\begin{aligned} \frac{de}{dt} &= -k_0 es + (k_0 + k_2)\, c, \\ \frac{ds}{dt} &= -k_1 es + k_0 c, \\ \frac{dc}{dt} &= k_1 es - (k_0 + k_2)\, c, \\ \frac{dp}{dt} &= k_2 c, \end{aligned}$$

where k_0, k_1, k_2 are rate constants. We impose initial conditions

$$s(0) = s_0, \quad e(0) = e_0, \quad c(0) = c_0, \quad p(0) = 0,$$

corresponding to an initial state with substrate and enzyme but no complex or product.

The equation for p decouples from the remaining equations. Adding the equations for e and c, we get

$$\frac{d}{dt}(e + c) = 0,$$

which implies that

$$e(t) + c(t) = e_0.$$

Thus, the equations reduce to a pair of ODEs for s and c:

$$\begin{aligned} \frac{ds}{dt} &= -k_1 e_0 s + (k_1 s + k_0)\, c, \\ \frac{dc}{dt} &= k_1 e_0 s - (k_1 s + k_0 + k_2)\, c, \\ s(0) &= s_0, \qquad c(0) = 0. \end{aligned}$$

We introduce dimensionless quantities

$$u(\tau) = \frac{s(t)}{s_0}, \quad v(\tau) = \frac{c(t)}{e_0}, \quad \tau = k_1 e_0 t,$$

$$\lambda = \frac{k_2}{k_1 s_0}, \quad k = \frac{k_0 + k_2}{k_1 s_0}, \quad \varepsilon = \frac{e_0}{s_0}.$$

Then u, v satisfy

$$\begin{aligned} &\frac{du}{d\tau} = -u + (u + k - \lambda)v, \\ &\varepsilon \frac{dv}{d\tau} = u - (u+k)v, \\ &u(0) = 1, \qquad v(0) = 0, \end{aligned} \tag{4.1}$$

where $\lambda > 0$ and $k > \lambda$.

The enzyme concentration is typically much less than the substrate concentration, and the ratio ε is usually in the range 10^{-2} to 10^{-7}. Thus, we want to solve (4.1) when ε is small.

This is a singular perturbation problem because the order of the system drops by one when $\varepsilon = 0$, and we cannot impose an initial condition on the dimensionless complex concentration v. As we will see below, what happens is this: there is an initial rapid adjustment of the complex and enzyme concentrations to quasi-equilibrium values on a time-scale of the order ε. Then there is a slower conversion of the substrate into the product on a time-scale of the order 1. We will construct inner and outer solutions that describe these processes and match them together.

4.1.1 *Outer solution*

We look for a straightforward expansion of the form

$$\begin{aligned} u(\tau, \varepsilon) &= u_0(\tau) + \varepsilon u_1(\tau) + O(\varepsilon^2), \\ v(\tau, \varepsilon) &= v_0(\tau) + \varepsilon v_1(\tau) + O(\varepsilon^2). \end{aligned}$$

Using these expansion in (4.1), and equating the leading order terms of the order ε^0, we find that

$$\begin{aligned} &\frac{du_0}{d\tau} = -u_0 + (u_0 + k - \lambda)\, v_0, \\ &0 = u_0 - (u_0 + k)\, v_0. \end{aligned}$$

We cannot impose both initial conditions on the leading-order outer solution. We will therefore take the most general solution of these equations. We will see, however, when we come to matching that the natural choice of imposing the initial condition $u_0(0) = 1$ is in fact correct.

From the second equation,

$$v_0 = \frac{u_0}{u_0 + k}.$$

This complex concentration v_0 corresponds to a quasi-equilibrium for the substrate concentration u_0, in which the creation of the complex by the binding of the enzyme with the substrate is balanced by the destruction of the complex by the reverse reaction and the decomposition of the complex into the product and the enzyme. Substituting this result into the first equation, we get a first order ODE for $u_0(\tau)$:

$$\frac{du_0}{d\tau} = -\frac{\lambda u_0}{u_0 + k}.$$

The solution of this equation is given by

$$u_0(\tau) + k \log u_0(\tau) = a - \lambda\tau,$$

where a is a constant of integration. This solution is invalid near $\tau = 0$ because no choice of a can satisfy the initial conditions for both u_0 and v_0.

4.1.2 *Inner solution*

There is a short initial layer, for times $t = O(\varepsilon)$, in which u, v adjust from their initial values to values that are compatible with the outer solution found above. We introduce inner variables

$$T = \frac{\tau}{\varepsilon}, \quad U(T, \varepsilon) = u(\tau, \varepsilon), \quad V(T, \varepsilon) = v(\tau, \varepsilon).$$

The inner equations are

$$\begin{aligned} &\frac{dU}{dT} = \varepsilon \left\{-U + (U + k - \lambda)V\right\}, \\ &\frac{dV}{dT} = U - (U + k)V, \\ &U(0, \varepsilon) = 1, \qquad V(0, \varepsilon) = 0. \end{aligned}$$

We look for an innner expansion

$$\begin{aligned} U(T, \varepsilon) &= U_0(T) + \varepsilon U_1(T) + O(\varepsilon^2), \\ V(T, \varepsilon) &= V_0(T) + \varepsilon V_1(T) + O(\varepsilon^2). \end{aligned}$$

The leading order inner equations are

$$\begin{aligned} &\frac{dU_0}{dT} = 0, \\ &\frac{dV_0}{dT} = U_0 - (U_0 + k)V_0, \\ &U_0(0) = 1, \qquad V_0(0) = 0. \end{aligned}$$

The solution is

$$\begin{aligned} U_0 &= 1, \\ V_0 &= \frac{1}{1 + k}\left[1 - e^{-(1+k)T}\right]. \end{aligned}$$

4.1.3 *Matching*

We assume that the inner and outer expansions are both valid for intermediate times of the order $\varepsilon \ll \tau \ll 1$. We require that the expansions agree asymptotically in this regime, where $T \to \infty$ and $\tau \to 0$ as $\varepsilon \to 0$. Hence, the matching condition is

$$\begin{aligned}\lim_{T\to\infty} U_0(T) &= \lim_{\tau\to 0^+} u_0(\tau), \\ \lim_{T\to\infty} V_0(T) &= \lim_{\tau\to 0^+} v_0(\tau).\end{aligned}$$

This condition implies that

$$u_0(0) = 1, \qquad v_0(0) = \frac{1}{1+k},$$

which is satisfied when $a = 1$ in the outer solution. Hence

$$u_0(\tau) + k \log u_0(\tau) = 1 - \lambda\tau.$$

The *slow manifold* for the enzyme system is the curve

$$v = \frac{u}{u+k}.$$

Trajectories rapidly approach the slow manifold in the initial layer. They then move more slowly along the slow manifold and approach the equilibrium $u = v = 0$ as $\tau \to \infty$. The inner layer corresponds to the small amount of enzyme 'loading up" on the substrate. The slow manifold corresponds to the enzyme working at full capacity in converting substrate into product.

A principle quantity of biological interest is the rate of uptake,

$$r_0 = \left.\frac{du_0}{d\tau}\right|_{\tau=0}.$$

It follows from the outer solution that

$$r_0 = \frac{\lambda}{1+k}.$$

The dimensional form of the rate of uptake is

$$\begin{aligned}R_0 &= \frac{ds}{dt} \\ &= \frac{Qs_0}{s_0 + k_m}\end{aligned}$$

where $Q = k_2 e_0$ is the maximum reaction rate, and

$$k_m = \frac{k_0 + k_2}{k_1}$$

is the Michaelis constant. The maximum rate depends only on k_2; the rate limiting step is $C \to P + E$.

For more information about enzyme kinetics, see [12].

4.2 General initial layer problems

Consider a dynamical system for $x(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^n$:

$$\begin{aligned} \dot{x} &= f(x, y), \\ \varepsilon \dot{y} &= g(x, y), \\ x(0) &= x_0, \qquad y(0) = y_0. \end{aligned} \tag{4.2}$$

Here, $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$. Suppose that there is a function $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$g\left(x, \varphi(x)\right) = 0,$$

and for each fixed $x \in \mathbb{R}^n$, the solution $y = \varphi(x)$ is a globally asymptotically stable equilibrium of the 'fast' system

$$\varepsilon \dot{y}(t) = g\left(x, y(t)\right). \tag{4.3}$$

Then the behavior of solutions of (4.2) is as follows:

(a) for $t = O(\varepsilon)$, there is a short initial layer in which $x(t)$ is nearly constant, and close to its initial value x_0, and $y(t)$ changes rapidly from its initial value to the quasi-steady state $y = \varphi(x_0)$.
(b) for $t = O(1)$, the solution is close to the slow manifold $y = \varphi(x) + O(\varepsilon)$, and $x(t)$ satisfies

$$\dot{x} = f\left(x, \varphi(x)\right).$$

If (4.3) does not have a unique globally stable equilibrium for every $x \in \mathbb{R}^m$, then more complex phenomena can occur.

An interesting example of a fast-slow system of ODEs arises in modeling the phenomenon of bursting in pancreatic β-cells. These cells are responsible for producing insulin which regulates glucose levels in the body. The β-cells are observed to undergo 'bursting' in which their membrane potential oscillates rapidly, with periods of the order of milliseconds. These oscillations stimulate the secretion of insulin by the cell. The length of each bursting period is on the order of seconds, and its length is influenced by the amount of glucose in the bloodstream. Thus, this mechanism provides one way that the body regulates glucose.

The basic mathematical model of bursting [12] consists of a fast/slow system. The fast system undergoes a Hopf bifurcation, corresponding to the appearance of a limit cycle oscillation, as the slow variable increases. On further increase in the slow variable the limit cycle disappears at a homoclinic bifurcation, and the fast system switches to a stable quasi-steady states. A decrease in the slow variable leads to a saddle-node bifurcation which destroys this quasi-steady state. When the fast system is in its limit-cycle state, it drives an increase in the slow variable, and when the fast system is in its quasi-steady state it drives a decrease in the slow variable. The overall effect of this dynamics is a periodic oscillation of the slow variable on a long time scale which switches on and off the rapid periodic bursting of the fast system.

4.3 Boundary layer problems

The following explicitly solvable model boundary-value problem for a second order linear ODE illustrates the phenomenon of boundary layers:

$$\begin{aligned} &\varepsilon y'' + 2y' + y = 0, \quad 0 < x < 1, \\ &y(0) = 0, \qquad y(1) = 1. \end{aligned} \tag{4.4}$$

Here, the prime denotes a derivative with respect to x, and ε is a small positive parameter. The order of the ODE reduces from two to one when $\varepsilon = 0$, so we cannot expect to impose both boundary conditions on the solution. As we will see, when ε is small, there is a thin boundary layer (of width the order of ε) near $x = 0$ where the solution changes rapidly to take on the boundary value.

4.3.1 *Exact solution*

The exponential solutions of this equation are $y = e^{mx}$ where

$$m = \frac{-1 \pm \sqrt{1-\varepsilon}}{\varepsilon}.$$

We write these roots as $m = -\alpha, -\beta/\varepsilon$ where

$$\begin{aligned} \alpha(\varepsilon) &= \frac{1 - \sqrt{1-\varepsilon}}{\varepsilon} \\ &= \frac{1}{2} + O(\varepsilon), \\ \beta(\varepsilon) &= 1 + \sqrt{1-\varepsilon} \\ &= 2 + O(\varepsilon). \end{aligned}$$

The general solution is

$$y(x, \varepsilon) = ae^{-\alpha(\varepsilon)x} + be^{-\beta(\varepsilon)x/\varepsilon}.$$

Imposing the boundary conditions and solving for the constants of integration a, b, we find that

$$y(x, \varepsilon) = \frac{e^{-\alpha x} - e^{-\beta x/\varepsilon}}{e^{-\alpha} - e^{-\beta/\varepsilon}}.$$

Thus, the solution involves two terms which vary on widely different length-scales.

Let us consider the behavior of this solution as $\varepsilon \to 0^+$. The asymptotic behavior is nonuniform, and there are two cases, which lead to matching outer and inner solutions.

(a) *Outer limit*: $x > 0$ fixed and $\varepsilon \to 0^+$. Then

$$y(x, \varepsilon) \to y_0(x),$$

where

$$y_0(x) = \frac{e^{-x/2}}{e^{-1/2}}. \tag{4.5}$$

This leading-order outer solution satisfies the boundary condition at $x = 1$ but not the boundary condition at $x = 0$. Instead, $y_0(0) = e^{1/2}$.

(b) *Inner limit*: $x/\varepsilon = X$ fixed and $\varepsilon \to 0^+$. Then

$$y(\varepsilon X, \varepsilon) \to Y_0(X),$$

where

$$Y_0(X) = \frac{1 - e^{-2X}}{e^{-1/2}}.$$

This leading-order inner solution satisfies the boundary condition at $x = 0$, but not the one at $x = 1$, which corresponds to $X = 1/\varepsilon$. Instead, we have $\lim_{X\to\infty} Y_0(X) = e^{1/2}$.

(c) *Matching*: Both the inner and outer expansions are valid in the region $\varepsilon \ll x \ll 1$, corresponding to $x \to 0$ and $X \to \infty$ as $\varepsilon \to 0$. They satisfy the matching condition

$$\lim_{x\to 0^+} y_0(x) = \lim_{X\to\infty} Y_0(X).$$

Let us construct an asymptotic solution of (4.4) without relying on the fact that we can solve it exactly.

4.3.2 *Outer expansion*

We begin with the outer solution. We look for a straightforward expansion

$$y(x, \varepsilon) = y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2).$$

We use this expansion in (4.4) and equate the coefficients of the leading-order terms to zero. Guided by our analysis of the exact solution, we only impose the boundary condition at $x = 1$. We will see later that matching is impossible if, instead, we attempt to impose the boundary condition at $x = 0$. We obtain that

$$2y_0' + y_0 = 0,$$
$$y_0(1) = 1.$$

The solution is given by (4.5), in agreement with the expansion of the exact solution.

4.3.3 *Inner expansion*

Next, we consider the inner solution. We suppose that there is a boundary layer at $x = 0$ of width $\delta(\varepsilon)$, and introduce a stretched inner variable $X = x/\delta$. We look for an inner solution

$$Y(X, \varepsilon) = y(x, \varepsilon).$$

Since

$$\frac{d}{dx} = \frac{1}{\delta}\frac{d}{dX},$$

we find from (4.4) that Y satisfies

$$\frac{\varepsilon}{\delta^2}Y'' + \frac{2}{\delta}Y' + Y = 0,$$

where the prime denotes a derivative with respect to X. There are two possible dominant balances in this equation: (a) $\delta = 1$, leading to the outer solution; (b) $\delta = \varepsilon$, leading to the inner solution. Thus, we conclude that the boundary layer thickness is of the order ε, and the appropriate inner variable is

$$X = \frac{x}{\varepsilon}.$$

The equation for Y is then

$$Y'' + 2Y' + \varepsilon Y = 0,$$
$$Y(0, \varepsilon) = 0.$$

We impose only the boundary condition at $X = 0$, since we do not expect the inner expansion to be valid outside the boundary layer where $x = O(\varepsilon)$.

We seek an inner expansion

$$Y(X, \varepsilon) = Y_0(X) + \varepsilon Y_1(X) + O(\varepsilon^2)$$

and find that

$$Y_0'' + 2Y_0' = 0,$$
$$Y_0(0) = 0.$$

The general solution of this problem is

$$Y_0(X) = c\left[1 - e^{-2X}\right], \tag{4.6}$$

where c is an arbitrary constant of integration.

4.3.4 *Matching*

We can determine the unknown constant c in (4.6) by requiring that the inner solution matches with the outer solution (4.5). Here the matching condition is simply that

$$\lim_{x\to 0^+} y_0(x) = \lim_{X\to\infty} Y_0(X),$$

which implies that

$$c = e^{1/2}.$$

In summary, the asymptotic solution as $\varepsilon \to 0^+$, is given by

$$y(x,\varepsilon) = \begin{cases} e^{1/2}\left[1 - e^{-2x/\varepsilon}\right] & \text{as } \varepsilon \to 0^+ \text{ with } x/\varepsilon \text{ fixed,} \\ e^{-x/2+1/2} & \text{as } \varepsilon \to 0^+ \text{ with } x > 0 \text{ fixed.} \end{cases}$$

A more systematic way to match solutions, which is useful in problems where the behavior of the solution is not as simple, is to introduce an intermediate variable $\xi = x/\eta(\varepsilon)$, where $\varepsilon \ll \eta(\varepsilon) \ll 1$ as $\varepsilon \to 0^+$, and require that the inner and outer solutions have the same asymptotic behavior as $\varepsilon \to 0^+$ with ξ fixed for suitably chosen η. This requirement holds provided that both the inner and outer expansions hold in an intermediate 'overlap' region in which $x = O(\eta)$.

4.3.5 *Uniform solution*

We have constructed two different inner and outer asymptotic solutions in two different regions. Sometimes it is convenient to have a single uniform solution. This can be constructed from the inner and outer solutions as follows:

$$y_{\text{uniform}} = y_{\text{inner}} + y_{\text{outer}} - y_{\text{overlap}}.$$

Here, the function y_{overlap} is the common asymptotic behavior of the inner and outer solutions in the matching region. Inside the boundary layer, we have $y_{\text{outer}} \sim y_{\text{overlap}}$, so $y_{\text{uniform}} \sim y_{\text{inner}}$. Away from the boundary layer, we have $y_{\text{inner}} \sim y_{\text{overlap}}$, so $y_{\text{uniform}} \sim y_{\text{outer}}$. Thus, in either case the uniform solution y_{uniform} has the correct asymptotic behavior.

For the model ODE problem solved above, we have $y_{\text{overlap}} = e^{1/2}$, and the leading order uniform solution is given by

$$y_{\text{uniform}}(x,\varepsilon) = e^{1/2}\left[e^{-x/2} - e^{-2x/\varepsilon}\right].$$

There are systematic matching methods that provide higher-order matched asymptotic solutions, but we will not discuss them here. In general such expansions may not converge, reflecting the singular nature of the perturbation problem. This can also be anticipated from the fact that the location of the boundary layer switches abruptly from $x = 0$ to $x = 1$ as the sign of ε switches from positive to negative.

4.3.6 *Why is the boundary layer at $x = 0$?*

Suppose we impose the boundary condition at $x = 0$ on the outer solution and look for an inner solution and a boundary layer at $x = 1$. The leading-order outer solution y_0 satisfies

$$\begin{aligned} 2y_0' + y_0 &= 0, \\ y_0(0) &= 0, \end{aligned}$$

so that

$$y_0(x) = 0.$$

We look for an inner expansion $y(x, \varepsilon) = Y(X, \varepsilon)$ in a boundary layer near $x = 1$, depending on a stretched inner variable

$$X = \frac{1-x}{\varepsilon}.$$

The leading-order inner solution $Y_0(X) = Y(X, 0)$ satisfies

$$\begin{aligned} Y_0'' - 2Y_0' &= 0, \\ Y_0(0) &= 1. \end{aligned}$$

The solution is

$$Y_0(X) = e^{2X} + c.$$

In this case, the inner solution grows exponentially into to interior of the domain, and $Y_0(X) \to \infty$ as $X \to \infty$. Thus, no matching with the outer solution is possible.

4.4 Boundary layer problems for linear ODEs

Consider the linear BVP

$$\begin{aligned} &\varepsilon y'' + a(x)y' + b(x)y = 0 \qquad 0 < x < 1, \\ &y(0) = \alpha, \qquad y(1) = \beta, \end{aligned}$$

where $a, b : [0, 1] \to \mathbb{R}$ are continuous functions, and α, β are constants.

The requirement that the inner, boundary layer solution decays exponentially into the interior of the interval implies that if $a(0) > 0$, then a boundary layer can occur at $x = 0$, and if $a(1) < 0$, then a boundary layer can occur at $x = 1$. Thus, if a does not change sign on $[0, 1]$, the boundary layer can occur at only one end, while if a changes sign, then more complicated behavior is possible:

(a) if $a(x) > 0$, then the boundary layer is at $x = 0$;
(b) if $a(x) < 0$, then the boundary layer is at $x = 1$;
(c) if $a(x)$ changes sign and $a'(x) > 0$, then a boundary layer cannot occur at either endpoint (in this case a corner layer typically occurs in the interior);
(d) if $a(x)$ changes sign and $a'(x) < 0$, then a boundary layer can occur at both endpoints.

The first two cases are treated by a straightforward modification of the expansion for constant coefficients. The other two cases are more difficult, and we illustrate them with some examples.

Example 4.1 Consider the BVP

$$\begin{aligned} &\varepsilon y'' + xy' - y = 0 \quad -1 < x < 1, \\ &y(-1) = 1, \qquad y(1) = 2. \end{aligned}$$

This ODE can be solved exactly. One solution is $y(x) = x$. A second linearly independent solution can be found by reduction of order, which gives

$$y(x) = e^{-x^2/(2\varepsilon)} + \frac{x}{\varepsilon} \int e^{-x^2/(2\varepsilon)}\, dx.$$

We will use the MMAE to construct an asymptotic solution without relying on an exact solution.

The inner solution grows exponentially into the interior at either end, so we cannot construct a boundary layer solution. We use instead left and right outer solutions

$$y(x, \varepsilon) = y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2),$$

where

$$xy_0' - y_0 = 0.$$

As we will see, matching implies that the left and right outer solutions are valid in the intervals $(-1, 0)$ and $(0, 1)$, respectively. Imposing the boundary conditions at the left and right, we therefore get

$$y(x, \varepsilon) \sim \begin{cases} -x & \text{as } \varepsilon \to 0^+ \text{ with } -1 \le x < 0 \text{ fixed}, \\ 2x & \text{as } \varepsilon \to 0^+ \text{ with } 0 < x < 1 \text{ fixed}. \end{cases}$$

These outer solutions meet at $x = 0$, where the coefficient of y' in the ODE vanishes. The outer solution has a 'corner' at that point.

We seek an inner solution inside a corner layer about $x = 0$. To find the appropriate scalings, we introduce the inner variables

$$X = \frac{x}{\delta}, \qquad Y = \eta Y,$$

and use a dominant balance argument. The rescaled ODE is

$$\frac{\varepsilon}{\delta^2}Y'' + XY' - Y = 0.$$

The dominant balance for the inner solution occurs when $\delta = \varepsilon^{1/2}$, and all three terms are of the same order of magnitude. Matching the inner solution with the left and right outer solutions, we find that

$$\eta Y(X, \varepsilon) \sim \begin{cases} -\delta X & \text{as } X \to -\infty, \\ 2\delta X & \text{as } X \to \infty. \end{cases}$$

We therefore choose $\eta = \delta$.

The leading order inner solution is then given by

$$y(x, \varepsilon) \sim \varepsilon^{1/2} Y_0\left(\frac{x}{\varepsilon^{1/2}}\right),$$

where $Y_0(X)$ satisfies

$$Y_0'' + XY_0' - Y_0 = 0,$$
$$Y_0(X) \sim \begin{cases} -X & \text{as } X \to -\infty, \\ 2X & \text{as } X \to \infty. \end{cases}$$

In this case, the ODE does not simplify at all; however, we obtain a canonical boundary value problem on $\mathbb{R}$ for matching the two outer solutions.

The solution of this inner problem is

$$Y_0(X) = -X + \frac{3}{\sqrt{2\pi}}\left[e^{-X^2/2} + X\int_{-\infty}^{X} e^{-t^2/2}\,dt\right],$$

and this completes the construction of the leading order asymptotic solution. (Other problems may lead to ODEs that require the use of special functions.)

Example 4.2 Consider the BVP

$$\varepsilon y'' - xy' + y = 0 \quad -1 < x < 1,$$
$$y(-1) = 1, \qquad y(1) = 2.$$

The coefficients of y and y' have the opposite sign to the previous example, and we can find an inner, boundary layer solution at both $x = 0$ and $x = 1$.

The leading order outer solution $y(x, \varepsilon) \sim y_0(x)$ satisfies

$$-xy_0' + y_0 = 0,$$

with solution

$$y_0(x) = Cx,$$

where C is a constant of integration.

The inner solution near $x = -1$ is given by

$$y(x, \varepsilon) = Y\left(\frac{1+x}{\varepsilon}, \varepsilon\right),$$

where $Y(X, \varepsilon)$ satisfies

$$Y'' + (1 - \varepsilon X)Y' + \varepsilon Y = 0,$$
$$Y(0, \varepsilon) = 1.$$

Expanding

$$Y(X, \varepsilon) = Y_0(X) + \varepsilon Y_1(X) + \ldots,$$

we find that the leading order inner solution $Y_0(X)$ satisfies

$$Y_0'' + Y_0' = 0,$$
$$Y_0(0) = 1.$$

The solution is

$$Y_0(X) = 1 + A\left(1 - e^{-X}\right),$$

where A is a constant of integration.

The inner solution near $x = 1$ is given by

$$y(x, \varepsilon) = Z\left(\frac{1-x}{\varepsilon}, \varepsilon\right),$$

where $Z(X, \varepsilon)$ satisfies

$$Z'' + (1 - \varepsilon X)Z' + \varepsilon Z = 0,$$
$$Z(0, \varepsilon) = 2.$$

Expanding

$$Z(X, \varepsilon) = Z_0(X) + \varepsilon Z_1(X) + \ldots,$$

we find that the leading order inner solution $Z_0(X)$ satisfies

$$Z_0'' + Z_0' = 0,$$
$$Z_0(0) = 2.$$

The solution is

$$Z_0(X) = 2 + B\left(1 - e^{-X}\right),$$

where B is a constant of integration.

The leading order matching condition implies that

$$\lim_{X\to\infty} Y_0(X) = \lim_{x\to -1} y_0(x),$$
$$\lim_{X\to\infty} Z_0(X) = \lim_{x\to 1} y_0(x),$$

or

$$1 + A = -C, \qquad 2 + B = C.$$

We conclude that

$$A = -(1 + C), \qquad B = C - 2.$$

The constant C is not determined by the matching conditions. Higher order matching conditions also do not determine C. Its value ($C = 1/2$) depends on the interaction between the solutions in the boundary layers at either end, which involves exponentially small effects [13].

4.5 A boundary layer problem for capillary tubes

In view of the subtle boundary layer behavior that can occur for linear ODEs, it is not surprising that the solutions of nonlinear ODEs can behave in even more complex ways. Various nonlinear boundary layer problems for ODEs are discussed in [3], [13], [19]. Here we will discuss a physical example of a boundary layer problem: the rise of a liquid in a wide capillary tube. This problem was first analyzed by Laplace [15]; see also [5], [14].

4.5.1 *Formulation*

Consider an open capillary tube of cross-section $\Omega \subset \mathbb{R}^2$ that is placed vertically in an infinite reservoir of fluid (such as water). Surface tension causes the fluid to rise up the tube, and we would like to compute the equilibrium shape of the meniscus and how high the fluid rises.

According the the Laplace-Young theory, there is a pressure jump $[p]$ across a fluid interface that is proportional to the mean curvature κ of the interface:

$$[p] = \sigma\kappa.$$

The constant of proportionality σ is the coefficient of surface tension.

We use (x, y) as horizontal coordinates and z as a vertical coordinate, where we measure the height z from the undisturbed level of the liquid far away from the tube and pressure p from the corresponding atmospheric pressure. Then, assuming that the fluid is in hydrostatic equilibrium, the pressure of a column of fluid of height z is $\rho g z$, where ρ is the density of the fluid (assumed constant), and g is the acceleration due to gravity.

If the fluid interface is a graph $z = u(x, y)$, then its mean curvature is given by

$$\kappa = -\nabla \cdot \left[\frac{\nabla u}{\left(1 + |\nabla u|^2\right)^{1/2}} \right],$$

where ∇ denotes the derivative with respect to the horizontal coordinates. Choosing the sign of the pressure jump appropriately, we find that u satisfies the following PDE in Ω

$$\sigma \nabla \cdot \left[\frac{\nabla u}{\left(1 + |\nabla u|^2\right)^{1/2}} \right] = \rho g u.$$

The boundary condition for the PDE follows from the fact that the fluid makes a fixed angle θ_w, called the wetting angle, with the wall of the tube. Hence on the boundary $\partial\Omega$, we have

$$\frac{\partial u}{\partial n} = \tan \theta_0,$$

where $\theta_0 = \pi/2 - \theta_w$. For definiteness, we assume that $0 < \theta_0 < \pi/2$.

Let a be a typical length-scale of the tube cross-section (for example, the radius of a circular tube). We introduce dimensionless variables

$$u^* = \frac{u}{a}, \quad x^* = \frac{x}{a}, \quad y^* = \frac{y}{a}.$$

Then, after dropping the stars, we find that the nondimensionalized problem is

$$\varepsilon^2 \nabla \cdot \left[\frac{\nabla u}{\left(1 + |\nabla u|^2\right)^{1/2}} \right] = u \qquad \text{in } \Omega,$$

$$\frac{\partial u}{\partial n} = \tan \theta_0 \qquad \text{on } \partial\Omega,$$

where

$$\varepsilon^2 = \frac{\sigma}{\rho g a^2}.$$

We define the capillary length-scale

$$\ell = \sqrt{\frac{\sigma}{\rho g}}.$$

This a characteristic length-scale for the balance between surface-tension and gravity forces, and we expect the fluid to rise up the tube by an amount of this order. We can write $\varepsilon = \ell/a$, meaning that it is the ratio of the capillary length-scale to the width of the tube. When $\varepsilon \gg 1$, we have a 'narrow' tube, and when $\varepsilon \ll 1$ we have a 'wide' tube.

4.5.2 *Wide circular tubes*

We now specialize to the case of a cylindrical tube with circular cross-section. In view of the symmetry of the problem, we expect that the height of the interface $z = u(r, \varepsilon)$ depends on the radial coordinate $r = (x^2 + y^2)^{1/2}$, and the PDE reduces to an ODE,

$$\frac{\varepsilon^2}{r}\left\{\frac{ru'}{[1+(u')^2]^{1/2}}\right\}' = u \qquad \text{in } 0 < r < 1,$$
$$u'(0) = 0, \qquad u'(1) = \tan\theta_0.$$

Here, the prime denotes a derivative with respect to r. The surface must have zero slope at $r = 0$ if it is to be smooth at the origin.

We will obtain an asymptotic solution for a wide circular tube, corresponding to the limit $\varepsilon \to 0$.* In this case, we expect that the fluid interface is almost flat over most of the interior of the tube (so that $u' \ll 1$, which linearizes the leading order equations), and rises near the boundary $r = 1$ to satisfy the boundary condition. We will obtain and match three leading-order asymptotic solutions:

(a) an inner solution valid near $r = 0$;
(b) an intermediate solution valid in $0 < r < 1$ (as we will see, this solution is the large-r limit of the inner solution, and matches with the boundary layer solution at $r = 1$);
(c) a boundary layer solution valid near $r = 1$.

Our main goal is to compute an asymptotic approximation as $\varepsilon \to 0$ for the height of the fluid at the center of the cylinder. The result is given in (4.13) below — the height is exponentially small in ε.

(a) The inner solution. We look for an inner solution near $r = 0$ of the form

$$u(r, \varepsilon) = \lambda U(R, \varepsilon), \qquad R = \frac{r}{\delta}, \tag{4.7}$$

where we will determine the scaling parameters $\lambda(\varepsilon)$, $\delta(\varepsilon)$ by matching and a dominant balance argument.

The slope u' of the interface is of the order

$$\alpha = \frac{\lambda}{\delta}.$$

Using (4.7) in the ODE, we get

$$\frac{\varepsilon^2}{\delta^2}\frac{1}{R}\left\{\frac{RU'}{[1+\alpha^2(U')^2]^{1/2}}\right\}' = U,$$

*An asymptotic solution can also be obtained for a narrow circular tube, an easier case since the problem is a regular perturbation problem as $\varepsilon \to \infty$.

where the prime denotes a derivative with respect to R. The dominant balance is $\delta = \varepsilon$, so the inner solution holds in a region of radius of the order ε about the origin.

The inner equation is then

$$\frac{1}{R}\left\{\frac{RU'}{[1+\alpha^2(U')^2]^{1/2}}\right\}' = U.$$

Since we expect that the interface is almost flat in the interior, we assume that $\alpha = o(1)$ as $\varepsilon \to 0$. (This assumption is consistent with the final solution, in which λ is exponentially small in ε.)

The leading order inner solution $U(R,\varepsilon) \sim U_0(R)$ then satisfies the linear equation

$$\frac{1}{R}\left(RU_0'\right)' = U_0,$$
$$U_0'(0) = 0.$$

We do not attempt to impose the boundary condition at $r = 1$, or $R = 1/\delta$, since we do not expect the inner solution to be valid in the boundary layer where u' is of the order one.

We choose the parameter λ in (4.7) so that $U_0(0) = 1$. Thus, to leading order in ε, λ is the height of the fluid at the center of the tube. It follows that

$$U_0(R) = I_0(R),$$

where I_0 is the modified Bessel function of order zero, which satisfies [17]

$$\frac{1}{R}\left(RI_0'\right)' - I_0 = 0,$$
$$I_0(0) = 1, \quad I_0'(0) = 0.$$

A power series expansion shows that there is a unique solution of this singular IVP.

The solution has the integral representation

$$I_0(R) = \frac{1}{\pi}\int_0^\pi e^{R\cos t}\,dt.$$

This function satisfies the initial conditions, and one can verify that it satisfies the ODE by direct computation and an integration by parts:

$$\begin{aligned}
\frac{1}{R}\left(RI_0'\right)' - I_0 &= I_0'' - I_0 + \frac{1}{R}I_0' \\
&= \frac{1}{\pi}\int_0^\pi \left(\cos^2 t - 1\right) e^{R\cos t}\,dt + \frac{1}{R\pi}\int_0^\pi \cos t\, e^{R\cos t}\,dt \\
&= -\frac{1}{\pi}\int_0^\pi \sin^2 t\, e^{R\cos t}\,dt + \frac{1}{\pi}\int_0^\pi (\sin t)'\,\frac{e^{R\cos t}}{R}\,dt \\
&= 0.
\end{aligned}$$

The asymptotic behavior of $I_0(R)$ as $R \to \infty$ can be computed from the integral representation by Laplace's method, and it grows exponentially in R. The phase $\cos t$ has a maximum at $t = 0$, and

$$\begin{aligned} I_0(R) &\sim \frac{1}{\pi} \int_0^\infty e^{R\left(1-\frac{1}{2}t^2\right)}\, dt \\ &\sim \frac{e^R}{\pi} \int_0^\infty e^{-Rt^2/2}\, dt \\ &\sim \frac{e^R}{\sqrt{2\pi R}} \end{aligned}$$

Hence, the outer expansion of this leading-order inner solution is

$$U_0(R) \sim \sqrt{\frac{e^R}{2\pi R}} \qquad \text{as } R \to \infty. \tag{4.8}$$

(b) Intermediate solution. In the region $0 < r < 1$, we expect that $u' \ll 1$. The leading order intermediate solution $u(r, \varepsilon) \sim u_0(r, \varepsilon)$ then satisfies

$$\frac{\varepsilon^2}{r} \left(r u_0'\right)' = u_0. \tag{4.9}$$

This is the same equation as the one for the inner solution, so the inner solution remains valid in this region. Nevertheless, it is instructive to obtain the asymptotic behavior of the solution directly from the ODE.

Away from $r = 0$, the solutions of (4.10) depend on two different length-scales: exponentially on a length-scale of the order ε and more slowly on a length-scale of the order one, arising from the dependence of the coefficients of the ODE on r due to the cylindrical geometry.

To account for this behavior, we use the WKB method, and look for solutions of the form

$$u_0(r, \varepsilon) = a(r, \varepsilon) e^{\varphi(r)/\varepsilon}. \tag{4.10}$$

One motivation for this form is that the constant-coefficients ODE obtained by 'freezing' the value of r at some nonzero constant value,

$$\varepsilon^2 u_0'' = u_0,$$

has solutions

$$u_0 = a e^{\pm r/\varepsilon},$$

where a is a constant. When the coefficients in the ODE depend upon r, we allow the amplitude a and the phase $\varphi(r) = \pm r$ to depend upon r in an appropriate way.

Using (4.10) in (4.9), and rewriting the result, we find that

$$a(\varphi')^2 + \varepsilon \left[2a'\varphi' + a\frac{1}{r}\left(r\varphi'\right)'\right] + \varepsilon^2 \frac{1}{r} \left(r a'\right)' = a.$$

We seek an asymptotic expansion of a,

$$a(r,\varepsilon) \sim a_0(r) + \varepsilon a_1(r) + \dots \qquad \text{as } \varepsilon \to 0.$$

Using this expansion in (4.5.2), expanding, and equating coefficients of ε^0 we find that

$$a_0\left[(\varphi')^2 - 1\right] = 0.$$

Hence, if $a_0 \neq 0$, we must have

$$(\varphi')^2 = 1.$$

Omitting the constants of integration, which can be absorbed into a, the solutions are

$$\varphi(r) = \pm r.$$

Equating coefficients of ε and simplifying the result, we find that

$$a_0' + \frac{1}{2r}a_0 = 0.$$

The solution is

$$a_0(r) = \frac{A}{r^{1/2}},$$

where A is a constant.

We therefore obtain that

$$u_0(r) \sim \frac{A_+}{r^{1/2}}e^{r/\varepsilon} + \frac{A_-}{r^{1/2}}e^{-r/\varepsilon}.$$

Matching this solution as $r \to 0^+$ with the the inner solution at $r = 0$, whose outer expansion is given in (4.8), and using $R = r/\varepsilon$, $U_0 = \lambda u_0$, we find that there are no terms that grow exponentially as $r \to 0^+$ so $A_- = 0$, and

$$A_+ = \lambda\sqrt{\frac{\varepsilon}{2\pi}}.$$

Thus, the outer expansion of the inner solution (4.8) is valid as $\varepsilon \to 0$ in the interior $0 < r < 1$, and the leading order behavior of the solution is given by

$$u(r,\varepsilon) \sim \lambda\sqrt{\frac{\varepsilon}{2\pi r}}e^{r/\varepsilon} \qquad \text{as } \varepsilon \to 0. \tag{4.11}$$

Here, the height $\lambda(\varepsilon)$ of the interface at the origin remains to be determined. We will find it by matching with the solution in the boundary layer.

(c) The boundary layer solution. Since $u'(1,\varepsilon) = \tan\theta_0 > 0$, we expect that the slope u' of the interface is of the order one in a boundary layer near $r = 1$. We therefore look for an inner boundary layer solution of the form

$$u(r,\varepsilon) = \delta U(X,\varepsilon), \qquad X = \frac{1-r}{\delta}.$$

A dominant balance argument gives $\delta = \varepsilon$, and then U satisfies the ODE

$$\frac{1}{1-\varepsilon X}\left\{\frac{(1-\varepsilon X)\,U'}{[1+(U')^2]^{1/2}}\right\}' = U,$$

where the prime denotes a derivative with respect to X. The boundary conditions are

$$U'(0,\varepsilon) = -\tan\theta_0, \qquad U'(X,\varepsilon) \to 0 \quad \text{as } X \to \infty.$$

The condition as $X \to \infty$ is a matching condition, which would need to be refined for higher-order approximations.

As $\varepsilon \to 0$, we have $U(X,\varepsilon) \sim U_0(X)$ where $U_0(X)$ satisfies the following BVP

$$\begin{aligned} &\left\{\frac{U_0'}{[1+(U_0')^2]^{1/2}}\right\}' = U_0 \qquad 0 < X < \infty, \\ &U_0'(0) = -\tan\theta_0, \\ &U_0'(X) \to 0 \quad \text{as } X \to \infty. \end{aligned}$$

The ODE is autonomous, corresponding to a two-dimensional planar problem, and (unlike the cylindrical problem) it can be solved exactly.

To solve the equation, it is convenient to introduce the angle $\psi > 0$ of the interface, defined by

$$\tan\psi = -U_0'. \tag{4.12}$$

We will use ψ as a new independent variable, and solve for $U_0 = U_0(\psi)$ and $X = X(\psi)$. The change of variables $X \mapsto \psi$ is well-defined if U_0' is strictly decreasing, as is the case, and then $X = 0$ corresponds to $\psi = \theta_0$ and $X = \infty$ corresponds to $\psi = 0$.

Differentiating (4.12) with respect to X, and writing X-derivatives as d/dX, we find that

$$\frac{d\psi}{dX} = -\frac{d^2U_0}{dX^2}\cos^2\psi.$$

The ODE implies that

$$\frac{d^2U_0/}{dX^2} = U_0 \sec^3\psi.$$

It follows that

$$\frac{d\psi}{dX} = -U_0 \sec\psi.$$

Using this result, the definition of ψ, and the equation

$$\frac{dU_0}{d\psi} = \frac{dU_0}{dX}\frac{dX}{d\psi},$$

we find that U_0, X satisfy the following ODEs:

$$\begin{aligned}\frac{dU_0}{d\psi} &= \frac{\sin\psi}{U_0},\\ \frac{dX}{d\psi} &= -\frac{\cos\psi}{U_0}.\end{aligned}$$

The boundary conditions on $X(\psi)$ are

$$X(\theta_0) = 0, \qquad X(\psi) \to \infty \quad \text{as } \psi \to 0^+.$$

The solution for U_0 is

$$U_0(\psi) = \sqrt{2(k - \cos\psi)},$$

where k is a constant of integration. The solution for X is

$$X(\psi) = \frac{1}{\sqrt{2}}\int_\psi^{\theta_0} \frac{\cos t}{\sqrt{k - \cos t}}\,dt,$$

where we have imposed the boundary condition $X(\theta_0) = 0$. The boundary condition that $X \to \infty$ as $\psi \to 0$ implies that $k = 1$, and then

$$U_0(\psi) = 2\sin\frac{\psi}{2}, \qquad X(\psi) = \frac{1}{2}\int_\psi^{\theta_0} \frac{\cos t}{\sin\frac{t}{2}}\,dt.$$

Evaluating the integral for X, we get

$$\begin{aligned}X(\psi) &= \frac{1}{2}\int_\psi^{\theta_0}\left(\operatorname{cosec}\frac{t}{2} - \sin\frac{t}{2}\right)dt\\ &= \left[\log\tan\frac{t}{4} + 2\cos\frac{t}{2}\right]_\psi^{\theta_0}\\ &= \log\tan\frac{\theta_0}{4} + 2\cos\frac{\theta_0}{2} - \log\tan\frac{\psi}{4} - 2\cos\frac{\psi}{2}.\end{aligned}$$

The asymptotic behaviors of U_0 and X as $\psi \to 0^+$ are given by

$$\begin{aligned}U_0(\psi) &= \psi + o(1),\\ X(\psi) &= \log\tan\frac{\theta_0}{4} + 2\cos\frac{\theta_0}{2} - \log\frac{\psi}{4} - 2 + o(1).\end{aligned}$$

It follows that the outer expansion of the leading-order boundary layer solution is

$$U_0(X) \sim 4\tan\left(\frac{\theta_0}{4}\right) e^{-4\sin^2(\theta_0/4)} e^{-X} \qquad \text{as } X \to \infty.$$

Rewriting this expansion in terms of the original variables, $u \sim \varepsilon U_0$, $r = 1 - \varepsilon X$, we get

$$u(r,\varepsilon) \sim 4\varepsilon \tan\left(\frac{\theta_0}{4}\right) e^{-4\sin^2(\theta_0/4)} e^{-1/\varepsilon} e^{r/\varepsilon}.$$

The inner expansion as $r \to 1^-$ of the leading order intermediate solution in (4.11) is

$$u(r,\varepsilon) \sim \lambda \sqrt{\frac{\varepsilon}{2\pi}} e^{r/\varepsilon}.$$

These solutions match if

$$\lambda = 4\tan\left(\frac{\theta_0}{4}\right) e^{-4\sin^2(\theta_0/4)} \sqrt{2\pi\varepsilon}\, e^{-1/\varepsilon}.$$

Thus, we conclude that

$$u(0,\varepsilon) \sim 4\tan\left(\frac{\theta_0}{4}\right) e^{-4\sin^2(\theta_0/4)} \sqrt{2\pi\varepsilon}\, e^{-1/\varepsilon} \qquad \text{as } \varepsilon \to 0. \tag{4.13}$$