Chapter 5

Method of Multiple Scales: ODEs

The method of multiple scales is needed for problems in which the solutions depend simultaneously on widely different scales. A typical example is the modulation of an oscillatory solution over time-scales that are much greater than the period of the oscillations. We will begin by describing the Poincaré-Lindstedt method, which uses a 'strained' time coordinate to construct periodic solutions. We then describe the method of multiple scales.

5.1 Periodic solutions and the Poincaré-Lindstedt expansion

We begin by constructing asymptotic expansions of periodic solutions of ODEs. The first example, Duffing's equation, is a Hamiltonian system with a family of periodic solutions. The second example, van der Pol's equation, has an isolated limit cycle.

5.1.1 *Duffing's equation*

Consider an undamped nonlinear oscillator described by Duffing's equation

$$y'' + y + \varepsilon y^3 = 0,$$

where the prime denotes a derivative with respect to time t. We look for solutions $y(t,\varepsilon)$ that satisfy the initial conditions

$$y(0,\varepsilon) = 1, \qquad y'(0,\varepsilon) = 0.$$

We look for straightforward expansion of an asymptotic solution as $\varepsilon \to 0$,

$$y(t,\varepsilon) = y_0(t) + \varepsilon y_1(t) + O(\varepsilon^2).$$

The leading-order perturbation equations are

$$\begin{aligned} &y_0'' + y_0 = 0, \\ &y_0(0) = 1, \quad y_0'(0) = 0, \end{aligned}$$

with the solution

$$y_0(t) = \cos t.$$

The next-order perturbation equations are

$$\begin{aligned} &y_1'' + y_1 + y_0^3 = 0, \\ &y_1(0) = 0, \quad y_1'(0) = 0, \end{aligned}$$

with the solution

$$y_1(t) = \frac{1}{32}\left[\cos 3t - \cos t\right] - \frac{3}{8} t \sin t.$$

This solution contains a *secular term* that grows linearly in t. As a result, the expansion is not uniformly valid in t, and breaks down when $t = O(\varepsilon)$ and εy_1 is no longer a small correction to y_0.

The solution is, in fact, a periodic function of t. The straightforward expansion breaks down because it does not account for the dependence of the period of the solution on ε. The following example illustrates the difficulty.

Example 5.1 We have the following Taylor expansion as $\varepsilon \to 0$:

$$\cos\left[(1+\varepsilon)t\right] = \cos t - \varepsilon t \sin t + O(\varepsilon^2).$$

This asymptotic expansion is valid only when $t \ll 1/\varepsilon$.

To construct a uniformly valid solution, we introduced a stretched time variable

$$\tau = \omega(\varepsilon) t,$$

and write $y = y(\tau, \varepsilon)$. We require that y is a 2π-periodic function of τ. The choice of 2π here is for convenience; any other constant period — for example 1 — would lead to the same asymptotic solution. The crucial point is that the period of y in τ is independent of ε (unlike the period of y in t).

Since $d/dt = \omega d/d\tau$, the function $y(\tau, \varepsilon)$ satisfies

$$\begin{aligned} &\omega^2 y'' + y + \varepsilon y^3 = 0, \\ &y(0, \varepsilon) = 1, \quad y'(0, \varepsilon) = 0, \\ &y(\tau + 2\pi, \varepsilon) = y(\tau, \varepsilon), \end{aligned}$$

where the prime denotes a derivative with respect to τ.

We look for an asymptotic expansion of the form

$$\begin{aligned} &y(\tau, \varepsilon) = y_0(\tau) + \varepsilon y_1(\tau) + O(\varepsilon^2), \\ &\omega(\varepsilon) = \omega_0 + \varepsilon \omega_1 + O(\varepsilon^2). \end{aligned}$$

Using this expansion in the equation and equating coefficients of ε^0, we find that

$$\begin{aligned}
&\omega_0^2 y_0'' + y_0 = 0, \\
&y_0(0) = 1, \quad y_0'(0) = 0, \\
&y_0(\tau + 2\pi) = y_0(\tau).
\end{aligned}$$

The solution is

$$\begin{aligned}
&y_0(\tau) = \cos\tau, \\
&\omega_0 = 1.
\end{aligned}$$

After setting $\omega_0 = 1$, we find that the next order perturbation equations are

$$\begin{aligned}
&y_1'' + y_1 + 2\omega_1 y_0'' + y_0^3 = 0, \\
&y_1(0) = 0, \quad y_1'(0) = 0, \\
&y_1(\tau + 2\pi) = y_1(\tau).
\end{aligned}$$

Using the solution for y_0 in the ODE for y_1, we get

$$\begin{aligned}
y_1'' + y_1 &= 2\omega_1 \cos\tau - \cos^3\tau \\
&= \left(2\omega_1 - \frac{3}{4}\right)\cos\tau - \frac{1}{4}\cos 3\tau.
\end{aligned}$$

We only have a periodic solution if

$$\omega_1 = \frac{3}{8},$$

and then

$$y_1(t) = \frac{1}{32}\left[\cos 3\tau - \cos\tau\right].$$

It follows that

$$\begin{aligned}
&y = \cos\omega t + \frac{1}{32}\varepsilon\left[\cos 3\omega t - \cos\omega t\right] + O(\varepsilon^2), \\
&\omega = 1 + \frac{3}{8}\varepsilon + O(\varepsilon^2).
\end{aligned}$$

This expansion can be continued to arbitrary orders in ε.

The appearance of secular terms in the expansion is a consequence of the non-solvability of the perturbation equations for periodic solutions.

Proposition 5.2 Suppose that $f : \mathbb{T} \to \mathbb{R}$ is a smooth 2π-periodic function, where $\mathbb{T}$ is the circle of length 2π. The ODE

$$y'' + y = f$$

has a 2π-periodic solution if and only if

$$\int_{\mathbb{T}} f(t)\cos t\,dt = 0, \qquad \int_{\mathbb{T}} f(t)\sin t\,dt = 0.$$

Proof. Let $L^2(\mathbb{T})$ be the Hilbert space of 2π-periodic, real-valued functions with inner product

$$\langle y, z\rangle = \int_{\mathbb{T}} y(t)z(t)\,dt.$$

We write the ODE as

$$Ay = f,$$

where

$$A = \frac{d^2}{dt^2} + 1.$$

Two integration by parts imply that

$$\begin{aligned}
\langle y, Az\rangle &= \int_{\mathbb{T}} y\left(z'' + z\right)\,dt \\
&= \int_{\mathbb{T}} \left(y'' + y\right) z\,dt \\
&= \langle Ay, z\rangle,
\end{aligned}$$

meaning that operator A is formally self-adjoint in $L^2(\mathbb{T})$. Hence, it follows that if $Ay = f$ and $Az = 0$, then

$$\begin{aligned}
\langle f, z\rangle &= \langle Ay, z\rangle \\
&= \langle y, Az\rangle \\
&= 0.
\end{aligned}$$

The null-space of A is spanned by $\cos t$ and $\sin t$. Thus, the stated condition is necessary for the existence of a solution.

When these solvability conditions hold, the method of variation of parameters can be used to construct a periodic solution

$$y(t) =$$

Thus, the conditions are also sufficient. □

In the equation for y_1, after replacing τ by t, we had

$$f(t) = 2\omega_1 \cos t - \cos 3t.$$

This function is orthogonal to $\sin t$, and

$$\langle f, \cos t\rangle = 2\pi \left\{2\omega_1 \overline{\cos^2 t} - \overline{\cos^4 t}\right\},$$

where the overline denotes the average value,

$$\overline{f} = \frac{1}{2\pi} \int_{\mathbb{T}} f(t)\, dt.$$

Since

$$\overline{\cos^2 t} = \frac{1}{2}, \qquad \overline{\cos^4 t} = \frac{3}{8},$$

the solvability condition implies that $\omega_1 = 3/8$.

5.1.2 *Van der Pol oscillator*

We will compute the amplitude of the limit cycle of the van der Pol equation with small damping,

$$y'' + \varepsilon \left(y^2 - 1\right) y' + y = 0.$$

This ODE describes a self-excited oscillator, whose energy increases when $|y| < 1$ and decreases when $|y| > 1$. It was proposed by van der Pol as a simple model of a beating heart. The ODE has a single stable periodic orbit, or limit cycle.

We have to determine both the period $T(\varepsilon)$ and the amplitude $a(\varepsilon)$ of the limit cycle. Since the ODE is autonomous, we can make a time-shift so that $y'(0) = 0$. Thus, we want to solve the ODE subject to the conditions that

$$\begin{aligned}
&y(t + T, \varepsilon) = y(t, \varepsilon),\\
&y(0, \varepsilon) = a(\varepsilon),\\
&y'(0, \varepsilon) = 0.
\end{aligned}$$

Using the Poincaré-Lindstedt method, we introduce a strained variable

$$\tau = \omega t,$$

and look for a 2π-periodic solution $y(\tau, \varepsilon)$, where $\omega = 2\pi/T$. Since $d/dt = \omega d/d\tau$, we have

$$\begin{aligned}
&\omega^2 y'' + \varepsilon\omega \left(y^2 - 1\right) y' + y = 0,\\
&y(\tau + 2\pi, \varepsilon) = y(\tau, \varepsilon),\\
&y(0, \varepsilon) = a,\\
&y'(0, \varepsilon) = 0,
\end{aligned}$$

where the prime denotes a derivative with respect to τ We look for asymptotic expansions,

$$\begin{aligned}
&y(\tau, \varepsilon) = y_0(\tau) + \varepsilon y_1(\tau) + O(\varepsilon^2),\\
&\omega(\varepsilon) = \omega_0 + \varepsilon\omega_1 + O(\varepsilon^2),\\
&a(\varepsilon) = a_0 + \varepsilon a_1 + O(\varepsilon^2).
\end{aligned}$$

Using these expansions in the equation and equating coefficients of ε^0, we find that

$$\begin{aligned}
&\omega_0^2 y_0'' + y_0 = 0,\\
&y_0(\tau + 2\pi) = y_0(\tau),\\
&y_0(0) = a_0,\\
&y_0'(0) = 0.
\end{aligned}$$

The solution is

$$\begin{aligned}
&y_0(\tau) = a_0 \cos\tau,\\
&\omega_0 = 1.
\end{aligned}$$

The next order perturbation equations are

$$\begin{aligned}
&y_1'' + y_1 + 2\omega_1 y_0'' + \left(y_0^2 - 1\right) y_0' = 0,\\
&y_1(\tau + 2\pi) = y_1(\tau),\\
&y_1(0) = a_1,\\
&y_1'(0) = 0.
\end{aligned}$$

Using the solution for y_0 in the ODE for y_1, we find that

$$y_1'' + y_1 = 2\omega_1 \cos\tau + a_0 \left(a_0^2 \cos^2\tau - 1\right) \sin\tau.$$

The solvability conditions, that the right and side is orthogonal to $\sin\tau$ and $\cos\tau$ imply that

$$\frac{1}{8}a_0^3 - \frac{1}{2}a_0 = 0, \quad \omega_1 = 0.$$

We take $a_0 = 2$; the solution $a_0 = -2$ corresponds to a phase shift in the limit cycle by π, and $a_0 = 0$ corresponds to the unstable steady solution $y = 0$. Then

$$y_1(\tau) = -\frac{1}{4}\sin 3\tau + \frac{3}{4}\sin\tau + \alpha_1 \cos\tau.$$

At the next order, in the equation for y_2, there are two free parameters, (a_1, ω_2), which can be chosen to satisfy the two solvability conditions. The expansion can be continued in the same way to all orders in ε.

5.2 The method of multiple scales

The Poincaré-Linstedt method provides a way to construct asymptotic approximations of periodic solutions, but it cannot be used to obtain solutions that evolve aperiodically on a slow time-scale. The method of multiple scales (MMS) is a more general approach in which we introduce one or more new ‘slow’ time variables for each time scale of interest in the problem. It does not require that the solution depends periodically on the ‘slow’ time variables.

We will illustrate the method by applying it to Mathieu's equation for $y(t, \varepsilon)$,

$$y'' + (1 + \delta + \varepsilon \cos kt)\, y = 0,$$

where (δ, ε, k) are constant parameters, and the prime denotes a derivative with respect to t. This equation describes a parametrically forced linear oscillator whose frequency is changed sinusoidally in time (e.g. small amplitude oscillations of a swing). The equilibrium $y = 0$ is unstable when the frequency k of the parametric forcing is sufficiently close to the resonant frequency $\sqrt{1 + \delta}$ of the unforced ($\varepsilon = 0$) oscillator.

We suppose that $\varepsilon \ll 1$ and

$$\delta = \varepsilon \delta_1,$$

where $\delta_1 = O(1)$ as $\varepsilon \to 0$. We will consider the case $k = 2$, which corresponds to the strongest instability, when $y(t, \varepsilon)$ satisfies

$$y'' + (1 + \varepsilon \delta_1 + \varepsilon \cos 2t)\, y = 0.$$

The idea of the MMS is to describe the evolution of the solution over long time-scales of the order ε^{-1} by the introduction of an additional 'slow' time variable

$$\tau = \varepsilon t.$$

We then look for a solution of the form

$$y(t, \varepsilon) = \tilde{y}(t, \varepsilon t, \varepsilon),$$

where $\tilde{y}(t, \tau, \varepsilon)$ is a function of two time variables (t, τ) that gives y when τ is evaluated at εt.

Applying the chain rule, we find that

$$\begin{aligned} y' &= \tilde{y}_t + \varepsilon \tilde{y}_\tau, \\ y'' &= \tilde{y}_{tt} + 2\varepsilon \tilde{y}_{t\tau} + \varepsilon^2 \tilde{y}_{\tau\tau}, \end{aligned}$$

where the subscripts denote partial derivatives. Using this result in the original equation, and denoting partial derivatives by subscripts, we find that $\tilde{y}(t, \tau, \varepsilon)$ satisfies

$$\tilde{y}_{tt} + 2\varepsilon \tilde{y}_{t\tau} + \varepsilon^2 \tilde{y}_{\tau\tau} + (1 + \varepsilon \delta_1 + \varepsilon \cos 2t)\, \tilde{y} = 0.$$

In fact, $\tilde{y}(t, \tau, \varepsilon)$ only has to satisfy this equation when $\tau = \varepsilon t$, but we will require that it satisfies it for all (t, τ). This requirement implies that y satisfies the original ODE. We have therefore replaced an ODE for y by a PDE for $\tilde{y}$. At first sight, this may not appear to be an improvement, but as we shall see we can use the extra flexibility provided by the dependence of $\tilde{y}$ on two variables to obtain an asymptotic solution for y that is valid for long times of the order ε^{-1}. Specifically, we will require that $y(t, \tau, \varepsilon)$ is a periodic function of the 'fast' variable t. Moreover, we only need to solve ODEs in t to construct this asymptotic solution.

We expand

$$\tilde{y}(t, \tau, \varepsilon) = y_0(t, \tau) + \varepsilon y_1(t, \tau) + O(\varepsilon^2).$$

We use this expansion in the equation for $\tilde{y}$, and equate coefficients of ε^0 and ε to zero. We find that

$$\begin{aligned} & y_{0tt} + y_0 = 0, \\ & y_{1tt} + y_1 + 2y_{0t\tau} + (\delta_1 + \cos 2t)\, y_0 = 0. \end{aligned}$$

The solution of the first equation is

$$y_0(t, \tau) = A(\tau)e^{it} + \text{c.c.}$$

Here, it is convenient to use complex notation. The amplitude $A(\tau)$ is an arbitrary complex valued function of the 'slow' time, and c.c. denotes the complex conjugate of the preceding terms.

Using this solution in the second equation, and writing the cosine in terms of exponentials, we find that y_1 satisfies

$$\begin{aligned} y_{1tt} + y_1 &= -2iA_\tau e^{it} - A\,(\delta_1 + \cos 2t)\, e^{it} + \text{c.c.} \\ &= -\frac{1}{2}Ae^{3it} - \left(2iA_\tau + \delta_1 A + \frac{1}{2}A^*\right) e^{it} + \text{c.c.} \end{aligned}$$

Here, the star denotes a complex conjugate. The solution for y_1 is periodic in t, and does not contain secular terms in t, if and only if the coefficient of the resonant term e^{it} is zero, which implies that $A(\tau)$ satisfies the ODE

$$2iA_\tau + \delta_1 A + \frac{1}{2}A^* = 0.$$

Writing $A = u + iv$ in terms of its real and imaginary parts, we find that

$$\begin{pmatrix} u \\ v \end{pmatrix}_\tau = \begin{pmatrix} 0 & \delta_1/2 - 1/4 \\ -\delta_1/2 - 1/4 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

The solutions of this equation are proportional to $e^{\pm\lambda\tau}$ where

$$\lambda = \frac{1}{2}\sqrt{\frac{1}{4} - \delta_1^2}.$$

Thus, in the limit $\varepsilon \to 0$, the equilibrium $y = 0$ is unstable when $|\delta_1| < 1/2$, or

$$|\delta| < \frac{1}{2}|\varepsilon|.$$

5.3 The method of averaging

Consider a system of ODEs for $x(t) \in \mathbb{R}^n$ which can be written in the following standard form

$$x' = \varepsilon f(x, t, \varepsilon). \tag{5.1}$$

Here, $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ is a smooth function that is periodic in t. We assume the period is 2π for definiteness, so that

$$f(x, t + 2\pi, \varepsilon) = f(x, t, \varepsilon).$$

Many problems can be reduced to this standard form by an appropriate change of variables.

Example 5.3 Consider a perturbed simple harmonic oscillator

$$y'' + y = \varepsilon h(y, y', \varepsilon).$$

We rewrite this equation as a first-order system and remove the unperturbed dynamics by introducing new dependent variables $x = (x_1, x_2)$ defined by

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We find, after some calculations, that (x_1, x_2) satisfy the system

$$\begin{aligned} x_1' &= -\varepsilon h \left(x_1 \cos t + x_2 \sin t, -x_1 \sin t + x_2 \cos t, \varepsilon \right) \sin t, \\ x_2' &= \varepsilon h \left(x_1 \cos t + x_2 \sin t, -x_1 \sin t + x_2 \cos t, \varepsilon \right) \cos t, \end{aligned}$$

which is in standard periodic form.

Using the method of multiple scales, we seek an asymptotic solution of (5.1) depending on a 'fast' time variable t and a 'slow' time variable $\tau = \varepsilon t$:

$$x = x(t, \varepsilon t, \varepsilon).$$

We require that $x(t, \tau, \varepsilon)$ is a 2π-periodic function of t:

$$x(t + 2\pi, \tau, \varepsilon) = x(t, \tau, \varepsilon).$$

Then $x(t, \tau, \varepsilon)$ satisfies the PDE

$$x_t + \varepsilon x_\tau = f(x, t, \varepsilon).$$

We expand

$$x(t, \tau, \varepsilon) = x_0(t, \tau) + \varepsilon x_1(t, \tau) + O(\varepsilon^2).$$

At leading order, we find that

$$x_{0t} = 0.$$

It follows that $x_0 = x_0(\tau)$ is independent of t, which is trivially a 2π-periodic function of t. At the next order, we find that x_1 satisfies

$$x_{1t} + x_{0\tau} = f(x_0, t, 0), \tag{5.2}$$
$$x_1(t + 2\pi, \tau) = x_1(t, \tau).$$

The following solvability condition is immediate.

Proposition 5.4 Suppose $f : \mathbb{R} \to \mathbb{R}^n$ is a smooth, 2π-periodic function. Then the $n \times n$ system of ODEs for $x(t) \in \mathbb{R}^n$,

$$x' = f(t),$$

has a 2π-periodc solution if and only if

$$\frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt = 0.$$

Proof. The solution is

$$x(t) = x(0) + \int_0^t f(s)\,ds.$$

We have

$$x(t + 2\pi) - x(t) = \int_t^{t+2\pi} f(s)\,ds,$$

which is zero if and only if f has zero mean over a period. □

If this condition does not hold, then the solution of the ODE grows linearly in time at a rate equal to the mean on f over a period.

An application of this proposition to (5.2) shows that we have a periodic solution for x_1 if and only if x_0 satisfies the averaged ODEs

$$x_{0\tau} = \overline{f}(x_0),$$

where

$$\overline{f}(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x, t, 0)\,dt.$$

First we state the basic existence theorem for ODEs, which implies that the solution of (5.1) exists on a time imterval ofthe order ε^{-1}.

Theorem 5.5 Consider the IVP

$$x' = \varepsilon f(x, t),$$
$$x(0) = x_0,$$

where $f : \mathbb{R}^n \times \mathbb{T} \to \mathbb{R}^n$ is a Lipschitz continuous function of $x \in \mathbb{R}^n$ and a continuous function of $t \in \mathbb{T}$. For $R > 0$, let

$$B_R(x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < R\},$$

where $|\cdot|$ denotes the Euclidean norm,

$$|x| = \sum_{i=1}^{n} |x_i|^2.$$

Let

$$M = \sup_{x \in B_R(x_0), t \in \mathbb{T}} |f(x,t)|.$$

Then there is a unique solution of the IVP,

$$x : (-T/\varepsilon, T/\varepsilon) \to B_R(x_0) \subset \mathbb{R}^n$$

that exists for the time interval $|t| < T/\varepsilon$, where

$$T = \frac{R}{M}.$$

Theorem 5.6 (Krylov-Bogoliubov-Mitropolski) With the same notation as the previous theorem, there exists a unique solution

$$\overline{x} : (-T/\varepsilon, T/\varepsilon) \to B_R(x_0) \subset \mathbb{R}^n$$

of the averaged equation

$$\overline{x}' = \varepsilon \overline{f}(\overline{x}),$$
$$\overline{x}(0) = x_0,$$

where

$$\overline{f}(x) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x,t)\, dt.$$

Assume that $f : \mathbb{R}^n \times \mathbb{T} \to \mathbb{R}^n$ is continuously differentiable. Let $0 < \widetilde{R} < R$, and define

$$\widetilde{T} = \frac{\widetilde{R}}{\widetilde{M}}, \qquad \widetilde{M} = \sup_{x \in B_{\widetilde{R}}(x_0), t \in \mathbb{T}} |f(x,t)|.$$

Then there exist constants $\varepsilon_0 > 0$ and $C > 0$ such that for all $0 \leq \varepsilon \leq \varepsilon_0$

$$|x(t) - \overline{x}(t)| \leq C\varepsilon \qquad \text{for } |t| \leq \widetilde{T}/\varepsilon.$$

A more geometrical way to view these results is in terms of Poincaré return maps. We define the Poincaré map $P^\varepsilon(t_0) : \mathbb{R}^n \to \mathbb{R}^n$ for (5.1) as the 2π-solution map. That is, if $x(t)$ is the solution of (5.1) with the initial condition $x(t_0) = x_0$, then

$$P^\varepsilon(t_0)x_0 = x(t_0 + 2\pi).$$

The choice of t_0 is not essential here, since different choices of t_0 lead to equivalent Poincaré maps when f is a 2π-periodic function of t. Orbits of the Poincaré map consist of closely spaced points when ε is small, and they are approximated by the trajectories of the averaged equations for times $t = O(1/\varepsilon)$.

5.4 The WKB method for ODEs

Suppose that the frequency of a simple harmonic oscillator is changing slowly compared with a typical period of the oscillation. For example, consider small-amplitude oscillations of a pendulum with a slowly varying length. How does the amplitude of the oscillations change?

The ODE describing the oscillator is

$$y'' + \omega^2(\varepsilon t)y = 0,$$

where $y(t, \varepsilon)$ is the amplitude of the oscillator, and $\omega(\varepsilon t) > 0$ is the slowly varying frequency.

Following the method of multiple scales, we might try to introduce a slow time variable $\tau = \varepsilon t$, and seek an asymptotic solutions

$$y = y_0(t, \tau) + \varepsilon y_1(t, \tau) + O(\varepsilon^2).$$

Then we find that

$$\begin{aligned} y_{0tt} + \omega^2(\tau)y_0 &= 0, \\ y_0(0) = a, \quad y_0'(0) &= 0, \end{aligned}$$

with solution

$$y_0(t, \tau) = a \cos\left[\omega(\tau)t\right].$$

At next order, we find that

$$y_{1tt} + \omega^2 y_1 + 2y_{0t\tau} = 0,$$

or

$$y_{1tt} + \omega^2 y_1 = 2a\omega\omega_\tau t \cos \omega t.$$

We cannot avoid secular terms that invalidate the expansion when $t = O(1/\varepsilon)$. The defect of this solution is that its period as a function of the 'fast' variable t depends on the 'slow' variable τ.

Instead, we look for a solution of the form

$$\begin{aligned} y &= y(\theta, \tau, \varepsilon), \\ \theta &= \frac{1}{\varepsilon}\varphi(\varepsilon t), \qquad \tau = \varepsilon t, \end{aligned}$$

where we require y to be 2π-periodic function of the 'fast' variable θ,

$$y(\theta + 2\pi, \tau, \varepsilon) = y(\theta, \tau, \varepsilon).$$

The choice of 2π for the period is not essential; the important requirement is that the period is a constant that does not depend upon τ.

By the chain rule, we have

$$\frac{d}{dt} = \varphi_\tau \partial_\theta + \varepsilon \partial_\tau,$$

and

$$y'' = (\varphi_\tau)^2 y_{\theta\theta} + \varepsilon \left\{2\varphi_\tau y_{\theta\tau} + \varphi_{\tau\tau} y_\theta\right\} + \varepsilon^2 y_{\tau\tau}.$$

It follows that y satisfies the PDE

$$(\varphi_\tau)^2 y_{\theta\theta} + \omega^2 y + \varepsilon \left\{2\varphi_\tau y_{\theta\tau} + \varphi_{\tau\tau} y_\theta\right\} + \varepsilon^2 y_{\tau\tau} = 0.$$

We seek an expansion

$$y(\theta, \tau, \varepsilon) = y_0(\theta, \tau) + \varepsilon y_1(\theta, \tau) + O(\varepsilon^2).$$

Then

$$(\varphi_\tau)^2 y_{0\theta\theta} + \omega^2 y_0 = 0.$$

Imposing the requirement that y_0 is a 2π-periodic function of θ, we find that

$$(\varphi_\tau)^2 = \omega^2,$$

which is satisfied if

$$\varphi(\tau) = \int_0^\tau \omega(\sigma)\, d\sigma.$$

The solution for y_0 is then

$$y_0(\theta, \tau) = A(\tau)e^{i\theta} + \text{c.c.},$$

where it is convenient to use complex exponentials, $A(\tau)$ is an arbitrary complex-valued scalar, and c.c. denotes the complex conjugate of the preceding term.

At the next order, we find that

$$\omega^2 (y_{1\theta\theta} + y_1) + 2\omega y_{0\theta\tau} + \omega_\tau y_{0\theta} = 0.$$

Using the solution for y_0 is this equation, we find that

$$\omega^2 (y_{1\theta\theta} + y_1) + i (2\omega A_\tau + \omega_\tau A) e^{i\theta} + \text{c.c.} = 0.$$

The solution for y_1 is periodic in θ if and only if A satisfies

$$2\omega A_\tau + \omega_\tau A = 0.$$

It follows that

$$\left(\omega |A|^2\right)_\tau = 0,$$

so that

$$\omega |A|^2 = \text{constant}.$$

Thus, the amplitude of the oscillator is proportional to $\omega^{-1/2}$ as its frequency changes.

The energy E ofthe oscillator is given by

$$\begin{aligned} E &= \frac{1}{2} (y')^2 + \frac{1}{\omega^2} y^2 \\ &= \frac{1}{2} \omega^2 |A|^2. \end{aligned}$$

Thus, E/ω is constant. The quantity E/ω is called the action. It is an example of an adiabatic invariant.

The WKB method can also be used to obtain asymptotic approximations as $\varepsilon \to 0$ of ODEs of the form

$$\varepsilon^2 y'' + V(x) y = 0.$$

This form corresponds to a change of variables $t \mapsto x/\varepsilon$ in the previous equations. The corresponding WKB expansion is is

$$\begin{aligned} &y(x, \varepsilon) = A(x, \varepsilon) e^{iS(x)/\varepsilon}, \\ &A(x, \varepsilon) = A_0(x) + \varepsilon A_1(x) + \ldots. \end{aligned}$$

This expansion breaks down at turning points where $V(x) = 0$, and then one must use Airy functions (or other functions at degenerate turning points) to describe the asymptotic behavior of the solution.

5.5 Perturbations of completely integrable Hamiltonian systems

Consider a Hamiltonian system whose configuration is described by n angles $x \in \mathbb{T}^n$, where $\mathbb{T}^n$ is the n-dimensional torus, with corresponding momenta $p \in \mathbb{R}^n$. The Hamiltonian $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ gives the energy of the system. The motion is described by Hamilton's equations

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x}.$$

This is a $2n \times 2n$ system of ODEs for $x(t)$, $p(t)$.

Example 5.7 The simple pendulum has Hamiltonian

$$H(x,p) = \frac{1}{2}p^2 + 1 - \cos x.$$

A change of coordinates $(x,p) \mapsto (\tilde{x},\tilde{p})$ that preserves the form of Hamilton's equations (for any Hamiltonian function) is called a canonical change of coordinates. A Hamiltonian system is completely integrable if there exists a canonical change of coordinates $(x,p) \mapsto (\varphi, I)$ such that $H = H(I)$ is independent of the angles $\varphi \in \mathbb{T}^n$. In these action-angle coordinates, Hamilton's equations become

$$\frac{d\varphi}{dt} = \frac{\partial H}{\partial I}, \qquad \frac{dI}{dt} = 0.$$

Hence, the solutions are $I =$ constant and

$$\varphi(t) = \omega(I)t + \varphi_0,$$

where

$$\omega(I) = \frac{\partial H}{\partial I}.$$

If

$$H^\varepsilon(\varphi, I) = H_0(I) + \varepsilon H_1(\varphi, I)$$

is a perturbation of a completely integrable Hamiltonian, then Hamilton's equations have the form

$$\frac{d\varphi}{dt} = \omega_0(I) + \varepsilon f(\varphi, I),$$
$$\frac{dI}{dt} = \varepsilon g(\varphi, I),$$

where

$$\omega_0 = \frac{\partial H_0}{\partial I}, \qquad f = \frac{\partial H_1}{\partial I}, \qquad g = -\frac{\partial H_1}{\partial \varphi}.$$

The study of these problems using multi-phase averaging methods is very subtle (e.g. KAM theory).