Chapter 5

Homogenization Theory

Homogenization theory is concerned with the derivation of equations for averages of solutions of equations with rapidly varying coefficients. This problem arises in obtaining macroscopic, or 'homogenized' or 'effective', equations for systems with a fine microscopic structure. Our goal is to represent a complex, rapidly-varying medium by a slowly-varying medium in which the fine-scale structure is averaged out in an appropriate way.

We will consider the homogenization of second-order linear elliptic PDEs. This is a fundamental and physically important example, but similar ideas apply to many other types of linear and nonlinear PDEs, such as Hamilton-Jacobi equations and various kinds of time-dependent PDEs.

5.1 Equilibrium problems

Suppose that $u : \mathbb{R}^d \to \mathbb{R}$ is the density of a conserved quantity, with flux $\mathbf{q} : \mathbb{R}^d \to \mathbb{R}^d$ and source density $f : \mathbb{R}^d \to \mathbb{R}$. We suppose that the flux is linearly related to the density gradient, so that

$$\mathbf{q} = -A\nabla u,$$

where A is the *conductivity tensor*. Such a linear constitutive relation holds in the commonly occurring case of systems that are sufficiently close to equilibrium. Far from equilibrium, the flux may be a nonlinear function of the density gradient, but we will not consider that case here.

In general, the conductivity tensor A is symmetric and positive-definite. In an isotropic medium, we have $A = \alpha I$, where I denotes the identity tensor and $\alpha > 0$. If the medium is nonuniform, then A depends on space $x \in \mathbb{R}^d$. In component notation, we write $A = (a_{ij})$ as a matrix, where $1 \leq i, j \leq d$.

In equilibrium, the total flux of the conserved quantity out of any (smooth) region $\Omega \subset \mathbb{R}^d$ is equal to the generation of the conserved quantity by sources inside the region, so that

$$\int_{\partial\Omega} \mathbf{q} \cdot \mathbf{n} \, dS = \int_{\Omega} f \, dx,$$

where $\mathbf{n}$ is the unit outward normal vector on the boundary $\partial\Omega$ of Ω. It follows from this equation and the divergence theorem that

$$\int_{\Omega} (\nabla \cdot \mathbf{q} - f) \, dx = 0.$$

Since this equation holds for arbitrary regions Ω, we conclude that

$$\nabla \cdot \mathbf{q} = f.$$

Hence, the density u satisfies the elliptic equation

$$-\nabla \cdot (A \nabla u) = f.$$

The component form of this equation is

$$-\frac{\partial}{\partial x_i} \left[a_{ij}(x) \frac{\partial u}{\partial x_j} \right] = f(x),$$

where we use the summation convention.

Example 5.1 Some physical examples are the following.

(a) *Heat flow*: T = temperature, $\mathbf{q}$ = heat flux, and f = heat source density. Then $\mathbf{q} = -A\nabla T$ is Fourier's law.
(b) *Porous media*: p = pressure, and $\mathbf{v}$ = velocity. Here, the velocity is related to the pressure gradient by Darcy's law, $\mathbf{v} = -A\nabla p$, and for incompressible flows we have $\nabla \cdot \mathbf{v} = 0$, so $f = 0$.
(c) *Electrostatics*: $\mathbf{D}$ = electric induction, $\mathbf{E}$ = electric intensity, and ρ = charge density. The electric induction and intensity are related in a linear medium by

$$\mathbf{D} = \varepsilon \mathbf{E},$$

where ε is the permittivity tensor. According to Maxwell's equations, we have $\mathbf{E} = -\nabla\varphi$ where φ is the electric potential, and

$$\nabla \cdot \mathbf{D} = 4\pi\rho.$$

Suppose we have a composite medium with a periodic structure.* Let λ be a typical microscopic length scale of a single period cell and L a typical macroscopic

*Homogenization theory also applies to stationary random media, although the analysis is more difficult than the periodic case.

length scale of the medium. Homogenization theory applies when

$$\varepsilon = \frac{\lambda}{L} \ll 1.$$

If we non-dimensionalize space variables by L, then we have

$$A = A\left(\frac{x}{\varepsilon}\right),$$

where $A(y)$ is a periodic function of y.

In equilibrium, the microscopic solution u^ε satisfies

$$-\nabla \cdot \left[A\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right] = f(x).$$

As $\varepsilon \to 0$, we have $u^\varepsilon \rightharpoonup u$, where the macroscopic solution u satisfies an effective, or homogenized, equation of the form

$$-\nabla \cdot \left[A^h \nabla u^\varepsilon\right] = f(x).$$

We want to obtain an expression for the effective, or homogenized, conductivity tensor A^h. As we will see, A^h is *not* simply the average of $A(y)$.

Example 5.2 Consider a one-dimensional medium made up of two materials, one with very low conductivity and the other with very high conductivity. It is clear that the conductivity of the composite is limited by the conductivity of the low-conductivity material, and it cannot be equal to the mean conductivity. As we will show, the effective conductivity of the composite is the harmonic mean of the conductivities of the components.

Example 5.3 The effective conductivity of a multi-dimensional mixture of two composites depends on the geometry of the mixture. For example, the effective conductivity of a medium made up of periodic layers is different from that of a medium made up of a periodic array of spherical inclusions, even if the volume fractions of the layers and the spheres are the same.

5.2 Multiple scale expansion

We write the PDE in component form,

$$-\frac{\partial}{\partial x_i}\left[a_{ij}\left(x, \frac{x}{\varepsilon}\right)\frac{\partial u}{\partial x_j}\right] = f\left(x, \frac{x}{\varepsilon}\right),$$

where we sum over repeated indices. We assume that:

(a) $a_{ij}(x, y) = a_{ji}(x, y)$;

(b) there is $\gamma > 0$ such that for all $\xi \in \mathbb{R}^d$

$$a_{ij}(x,y)\xi_i\xi_j \geq \gamma\xi_i\xi_i,$$

meaning that the PDE is elliptic;

(c) $a_{ij}(x, y + e_k) = a_{ij}(x,y)$, where e_k, $k = 1, \ldots, d$, is the unit vector in the k direction, meaning that the coefficients are periodic in y with a cubic unit cell, so that $a_{ij}(x,\cdot) : \mathbb{T}^d \to \mathbb{R}$, where $\mathbb{T}^d$ is the d-dimensional unit torus (other unit cells can be treated in a similar way);

(d) $f(x,y)$ is periodic in y, so that $f(x,\cdot) : \mathbb{T}^d \to \mathbb{R}$.

We look for an asymptotic solution of the form

$$u(x,\varepsilon) = v\left(x, \frac{x}{\varepsilon}, \varepsilon\right),$$

for a suitable function $v(x,y,\varepsilon)$. Then

$$\frac{\partial u}{\partial x_i} = \frac{1}{\varepsilon}\frac{\partial v}{\partial y_i} + \frac{\partial v}{\partial x_i},$$

and using the method of multiple scales, we require that $v(x,y,\varepsilon)$ satisfies the PDE

$$-\frac{1}{\varepsilon^2}\frac{\partial}{\partial y_i}\left[a_{ij}(x,y)\frac{\partial v}{\partial y_j}\right] - \frac{1}{\varepsilon}\frac{\partial}{\partial x_i}\left[a_{ij}(x,y)\frac{\partial v}{\partial y_j}\right] - \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\left[a_{ij}(x,y)\frac{\partial v}{\partial x_j}\right] - \frac{\partial}{\partial x_i}\left[a_{ij}(x,y)\frac{\partial v}{\partial x_j}\right] = f(x,y).$$

We further require that $v(x,y,\varepsilon)$ is periodic in y, so that

$$v(x, y + e_k, \varepsilon) = v(x,y,\varepsilon) \qquad \text{for } k = 1, \ldots, d,$$

or $v(x,\cdot,\varepsilon) : \mathbb{T}^d \to \mathbb{R}$.

We seek an asymptotic expansion for v of the form

$$v(x,y,\varepsilon) = v_0(x,y) + \varepsilon v_1(x,y) + \varepsilon^2 v_2(x,y) + \ldots.$$

We use this expansion in the PDE and equate coefficients of powers of ε. At the order ε^{-2}, we find that v_0 satisfies

$$-\frac{\partial}{\partial y_i}\left[a_{ij}\frac{\partial v_0}{\partial y_j}\right] = 0.$$

Proposition 5.4 Suppose that $a_{ij} : \mathbb{T}^d \to \mathbb{R}$, where $a_{ij} \in L^\infty(\mathbb{T}^d)$ satisfies the ellipticity conditions stated above. If $v : \mathbb{T}^d \to \mathbb{R}$, where $v \in H^1(\mathbb{T}^d)$, is a periodic function that satisfies the elliptic PDE

$$-\frac{\partial}{\partial y_i}\left[a_{ij}\frac{\partial v}{\partial y_j}\right] = 0,$$

then $v = \text{constant}$.

Proof. We multiply the PDE by v, integrate over $\mathbb{T}^d$, and integrate by parts. The boundary terms vanish if the solution is periodic, and hence

$$\int_{\mathbb{T}^d} a_{ij} \frac{\partial v}{\partial y_i} \frac{\partial v}{\partial y_j} \, dy = 0.$$

Ellipticity implies that the integrand is non-negative, and therefore

$$a_{ij} \frac{\partial v}{\partial y_i} \frac{\partial v}{\partial y_j} = 0,$$

which implies that

$$\frac{\partial v}{\partial y_i} = 0,$$

so $v = \text{constant}$. □

Connected with this result is the following solvability condition.

Proposition 5.5 Suppose that $a_{ij} : \mathbb{T}^d \to \mathbb{R}$ and $f : \mathbb{T}^d \to \mathbb{R}$ are periodic functions, where $a_{ij} \in L^\infty(\mathbb{T})$ satisfies the ellipticity conditions stated above, and $f \in H^{-1}(\mathbb{T}^d)$. Let angular brackets denote an average with respect to y,

$$\langle f \rangle = \int_{\mathbb{T}^d} f(y) \, dy.$$

Then the PDE

$$-\frac{\partial}{\partial y_i} \left[a_{ij}(y) \frac{\partial v}{\partial y_j} \right] = f$$

has a periodic solution $v : \mathbb{T}^d \to \mathbb{R}$, with $v \in H^{-1}(\mathbb{T}^d)$, if and only if

$$\langle f \rangle = 0.$$

Proof. An integration by parts implies that the operator $L : H^1(\mathbb{T}^d) \to H^{-1}(\mathbb{T}^d)$ defined by

$$Lv = -\frac{\partial}{\partial y_i} \left[a_{ij} \frac{\partial v}{\partial y_j} \right]$$

is self-adjoint, meaning that

$$\langle u, Lv \rangle = \langle Lu, v \rangle$$

for all $u, v \in H^1(\mathbb{T}^d)$, where for $u, v \in L^2(\mathbb{T}^d)$

$$\langle u, v \rangle = \int_{\mathbb{T}^d} u(y) v(y) \, dy.$$

The null space of L is spanned by constant functions, and thus $Lu = f$ is solvable only if f is orthogonal to 1, which implies that its mean is zero. Equivalently, we can obtain this necessary solvability condition by averaging the PDE over a unit cell.

Conversely, if $\langle f \rangle = 0$, then the variational theory of elliptic PDEs implies that the PDE is solvable. The solution is unique up to an arbitrary additive constant$\square$

Since x occurs in the leading-order equation as a parameter, it follows that $v_0 = v_0(x)$. Thus, v_0 depends only on the 'slow' variable x and is perturbed by small rapidly-varying fluctuations.

At the order ε^{-1}, after using the fact that $v_0 = v_0(x)$, we find that v_1 satisfies

$$-\frac{\partial}{\partial y_i}\left[a_{ij}\frac{\partial v_1}{\partial y_j}\right] = \frac{\partial a_{ij}}{\partial y_i}\frac{\partial v_0}{\partial x_j}.$$

Let $w_k(x,\cdot) : \mathbb{T}^d \to \mathbb{R}$, for $k = 1, \dots, d$, be the solution of the 'cell-problem'

$$-\frac{\partial}{\partial y_i}\left[a_{ij}(x,y)\frac{\partial w_k}{\partial y_j}\right] = \frac{\partial a_{ik}}{\partial y_i}(x,y).$$

(To specify $w_k(x,y)$ uniquely, we can require that it has zero mean over y.) In general, we cannot compute w_k explicitly, but the solution exists by the previous proposition because $\partial a_{ik}/\partial y_i$ is a derivative and therefore has zero mean. By linearity, we find that

$$v_1(x,y) = \overline{v}_1(x) + \frac{\partial v_0}{\partial x_k}(x) w_k(x,y),$$

where $\overline{v}_1(x)$ is an arbitrary function of integration.

At the order 1, we find that v_2 satisfies

$$-\frac{\partial}{\partial y_i}\left(a_{ij}\frac{\partial v_2}{\partial y_j}\right) - \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial v_1}{\partial y_j}\right) - \frac{\partial}{\partial y_i}\left(a_{ij}\frac{\partial v_1}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial v_0}{\partial x_j}\right) = f.$$

Averaging this equation with respect to y, we obtain the solvability condition

$$-\frac{\partial}{\partial x_i}\left(\left\langle a_{ij}\frac{\partial v_1}{\partial y_j}\right\rangle\right) - \frac{\partial}{\partial x_i}\left(\langle a_{ij}\rangle \frac{\partial v_0}{\partial x_j}\right) = \langle f \rangle.$$

Using the expression for v_1 in this equation, we find that v_0 satisfies the homogenized equation

$$-\frac{\partial}{\partial x_i}\left(a_{ij}^h\frac{\partial v_0}{\partial x_j}\right) = \langle f \rangle,$$

where

$$a_{ij}^h = \left\langle a_{ij} + a_{ik}\frac{\partial w_j}{\partial y_k}\right\rangle.$$

The second term on the right hand side gives the correction between the effective conductivity a_{ij}^h and the mean conductivity $\langle a_{ij}\rangle$.

Example 5.6 In the case of one space dimension, where the conductivity is a scalar function $a(y)$ assumed independent of the 'slow' variable x for simplicity, the cell-problem is

$$-\frac{d}{dy}\left(a\frac{dw}{dy}\right) = \frac{da}{dy}.$$

Integration of this equation implies that

$$\frac{dw}{dy} = -1 + \frac{k}{a},$$

where k is a constant of integration. Averaging this equation over a period and using the periodicity of w, we find that

$$k = \frac{1}{\langle 1/a \rangle}.$$

The effective conductivity a^h is given by

$$\begin{aligned} a^h &= \left\langle a + a\frac{dw}{dy} \right\rangle \\ &= k. \end{aligned}$$

Hence, the effective conductivity is the harmonic mean of the microscopic conductivity.