## Chapter 3

# The Contraction Mapping Theorem

In this chapter we state and prove the *contraction mapping theorem*, which is one of the simplest and most useful methods for the construction of solutions of linear and nonlinear equations. We also present a number of applications of the theorem.

### 3.1 Contractions

**Definition 3.1** Let $(X, d)$ be a metric space. A mapping $T : X \to X$ is a *contraction mapping*, or *contraction*, if there exists a constant $c$, with $0 \leq c < 1$, such that

$$d(T(x), T(y)) \leq c\, d(x, y) \tag{3.1}$$

for all $x, y \in X$.

Thus, a contraction maps points closer together. In particular, for every $x \in X$, and any $r > 0$, all points $y$ in the ball $B_r(x)$, are mapped into a ball $B_s(Tx)$, with

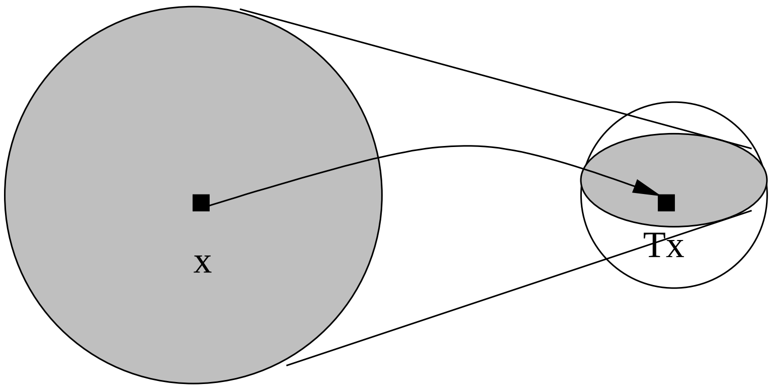


Fig. 3.1 $T$ is a contraction.

$s < r$. This is illustrated in Figure 3.1. Sometimes a map satisfying (3.1) with $c = 1$ is also called a contraction, and then a map satisfying (3.1) with $c < 1$ is called a *strict contraction*. It follows from (3.1) that a contraction mapping is uniformly continuous.

If $T : X \to X$, then a point $x \in X$ such that

$$T(x) = x \tag{3.2}$$

is called a *fixed point* of $T$. The contraction mapping theorem states that a strict contraction on a complete metric space has a unique fixed point. The contraction mapping theorem is only one example of what are more generally called fixed-point theorems. There are fixed-point theorems for maps satisfying (3.1) with $c = 1$, and even for arbitrary continuous maps on certain metric spaces. For example, the *Schauder fixed point theorem* states that a continuous mapping on a convex, compact subset of a Banach space has a fixed point. The proof is topological in nature (see Kantorovich and Akilov [27]), and we will not discuss such fixed point theorems in this book.

In general, the condition that $c$ is strictly less than one is needed for the uniqueness and the existence of a fixed point. For example, if $X = \{0, 1\}$ is the discrete metric space with metric determined by $d(0, 1) = 1$, then the map $T$ defined by $T(0) = 1$, $T(1) = 0$ satisfies (3.1) with $c = 1$, but $T$ does not have any fixed points. On the other hand, the identity map on any metric space satisfies (3.1) with $c = 1$, and every point is a fixed point.

It is worth noting that (3.2), and hence its solutions, do not depend on the metric $d$. Thus, if we can find any metric on $X$ such that $X$ is complete and $T$ is a contraction on $X$, then we obtain the existence and uniqueness of a fixed point. It may happen that $X$ is not complete in any of the metrics for which one can prove that $T$ is a contraction. This can be an indication that the solution of the fixed point problem does not belong to $X$, but to a larger space, namely the completion of $X$ with respect to a suitable metric $d$.

**Theorem 3.2 (Contraction mapping)** If $T : X \to X$ is a contraction mapping on a complete metric space $(X, d)$, then there is exactly one solution $x \in X$ of (3.2).

***Proof.*** The proof is constructive, meaning that we will explicitly construct a sequence converging to the fixed point. Let $x_0$ be any point in $X$. We define a sequence $(x_n)$ in $X$ by

$$x_{n+1} = Tx_n \qquad \text{for } n \geq 0.$$

To simplify the notation, we often omit the parentheses around the argument of a map. We denote the $n$th iterate of $T$ by $T^n$, so that $x_n = T^n x_0$.

First, we show that $(x_n)$ is a Cauchy sequence. If $n \geq m \geq 1$, then from (3.1) and the triangle inequality, we have

$$\begin{aligned}
d(x_n, x_m) &= d(T^n x_0, T^m x_0) \\
&\leq c^m d(T^{n-m} x_0, x_0) \\
&\leq c^m \left[ d(T^{n-m} x_0, T^{n-m-1} x_0) + d(T^{n-m-1} x_0, T^{n-m-2} x_0) \right. \\
&\qquad \left. + \cdots + d(T x_0, x_0) \right] \\
&\leq c^m \left[ \sum_{k=0}^{n-m-1} c^k \right] d(x_1, x_0) \\
&\leq c^m \left[ \sum_{k=0}^{\infty} c^k \right] d(x_1, x_0) \\
&\leq \left( \frac{c^m}{1-c} \right) d(x_1, x_0),
\end{aligned}$$

which implies that $(x_n)$ is Cauchy. Since $X$ is complete, $(x_n)$ converges to a limit $x \in X$. The fact that the limit $x$ is a fixed point of $T$ follows from the continuity of $T$:

$$Tx = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} T x_n = \lim_{n\to\infty} x_{n+1} = x.$$

Finally, if $x$ and $y$ are two fixed points, then

$$0 \leq d(x, y) = d(Tx, Ty) \leq c d(x, y).$$

Since $c < 1$, we have $d(x, y) = 0$, so $x = y$ and the fixed point is unique. □

## 3.2 Fixed points of dynamical systems

A dynamical system describes the evolution in time of the state of some system. Dynamical systems arise as models in many different disciplines, including physics, chemistry, engineering, biology, and economics. They also arise as an auxiliary tool for solving other problems in mathematics, and the properties of dynamical systems are of intrinsic mathematical interest.

A dynamical system is defined by a state space $X$, whose elements describe the different states the system can be in, and a prescription that relates the state $x_t \in X$ at time $t$ to the state at a previous time. We call a dynamical system continuous or discrete, depending on whether the time variable is continuous or discrete. For a continuous dynamical system, the time $t$ belongs to an interval in $\mathbb{R}$, and the dynamics of the system is typically described by an ODE of the form

$$\dot{x} = f(x), \tag{3.3}$$

where the dot denotes a time derivative, and $f$ is a vector field on $X$. There is little loss of generality in assuming this form of equation. For example, a second order, nonautonomous ODE

$$\ddot{y} = g(t, y, \dot{y})$$

may be written as a first order, autonomous system of the form (3.3) for the state variable $x = (s, y, v)$, where $s = t$ and $v = \dot{y}$, with

$$\dot{s} = 1, \qquad \dot{y} = v, \qquad \dot{v} = g(s, y, v).$$

For a discrete dynamical system, we may take time $t = n$ to be an integer, and the dynamics is defined by a map $T : X \to X$ that relates the state $x_{n+1}$ at time $t = n + 1$ to the state $x_n$ at time $t = n$,

$$x_{n+1} = Tx_n. \tag{3.4}$$

If $T$ is not invertible, then the dynamics is defined only forward in time, while if $T$ is invertible, then the dynamics is defined both backward and forward in time. A fixed point of the map $T$ correponds to an equilibrium state of the discrete dynamical system. If the state space is a complete metric space and $T$ is a contraction, then the contraction mapping theorem implies that there is a unique equilibrium state, and that the system approaches this state as time tends to infinity starting from any initial state. In this case, we say that the fixed point is globally asymptotically stable.

One of the simplest, and most famous, discrete dynamical systems is the *logistic equation* of population dynamics,

$$x_{n+1} = 4\mu x_n \left(1 - x_n\right), \tag{3.5}$$

where $0 \leq \mu \leq 1$ is a parameter, and $x_n \in [0, 1]$. This equation is of the form (3.4) where $T : [0, 1] \to [0, 1]$ is defined by

$$Tx = 4\mu x(1 - x).$$

See Figure 3.2 for a plot of $x \mapsto Tx$, for three different values of $\mu$. We may interpret $x_n$ as the population of the $n$th generation of a reproducing species. The linear equation $x_{n+1} = 4\mu x_n$ describes the exponential growth (if $4\mu > 1$) or decay (if $4\mu < 1$) of a population with constant birth or death rate. The nonlinearity in (3.5) provides a simple model for a species in which the effects of overcrowding lead to a decline in the birth rate as the population increases.

The logistic equation shows that the iterates of even very simple nonlinear maps can have amazingly complex behavior. Analyzing the full behavior of the logistic map is beyond the scope of the elementary application of the contraction mapping theorem we give here.

When $0 \leq \mu \leq 1/4$, the point 0 is the only fixed point of $T$ in $[0, 1]$, and $T$ is a contraction on $[0, 1]$. The proof of the contraction mapping theorem therefore

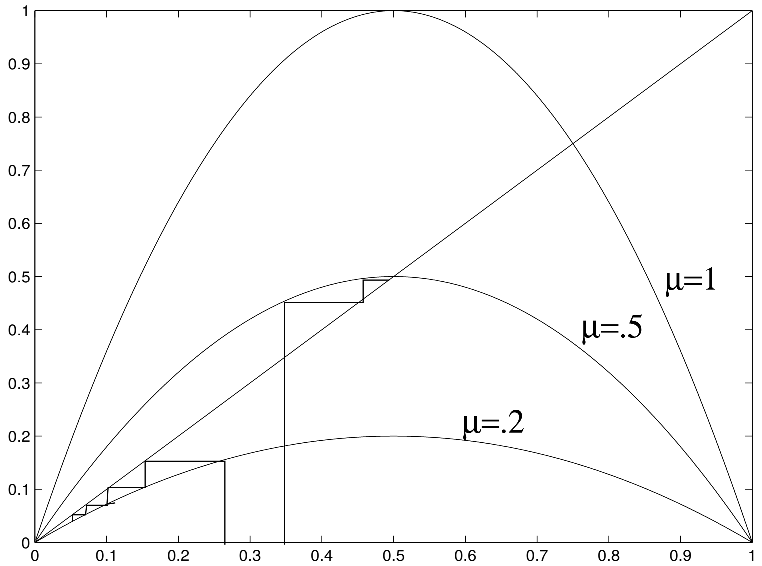


Fig. 3.2 The logistic map for three different values of the parameter $\mu$. For $\mu = 0.2$, 0 is the only fixed point and it is stable. For $\mu = 0.5$, 0 is an unstable fixed point and there is a nonzero, stable, fixed point. For $\mu = 1$, the two fixed points are unstable and more complex (and more interesting) asymptotic behavior of the dynamical system occurs.

implies that $T^n x_0 \to 0$ for an arbitrary initial population $x_0 \in [0, 1]$, meaning that the population dies out. When $1/4 < \mu \leq 1$, there is a second fixed point at $x = (4\mu - 1)/4\mu$. The appearance of a new fixed point as the parameter $\mu$ varies is an example of a *bifurcation* of fixed points. As $\mu$ increases further, there is an infinite sequence of more complicated bifurcations, leading to chaotic dynamics for $\mu \geq 0.89\ldots$.

As a second application of the contraction mapping theorem, we consider the solution of an equation $f(x) = 0$. One way to obtain a solution is to recast the equation in the form of a fixed point equation $x = Tx$, and then construct approximations $x_n$ starting from an initial guess $x_0$ by the iteration scheme

$$x_{n+1} = Tx_n.$$

In other words, we are attempting to find the solution as the time-asymptotic state of an associated discrete dynamical system which has the solution as a stable fixed point. Similar ideas apply in other contexts. For example, we may attempt to construct the solution of an elliptic PDE as the time-asymptotic state of an associated parabolic PDE.

There are many ways to rewrite an equation $f(x) = 0$ as a fixed point problem, some of which will work better than others. Ideally, we would like to rewrite the equation as a fixed point equation in which $T$ is a contraction on the whole space, or at least a contraction on some set that contains the solution we seek.

To provide a simple illustration of these ideas, we prove the convergence of an algorithm to compute square roots. If $a > 0$, then $x = \sqrt{a}$ is the positive solution of the equation

$$x^2 - a = 0.$$

We rewrite this equation as the fixed point problem

$$x = \frac{1}{2}\left(x + \frac{a}{x}\right).$$

The associated iteration scheme is then

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right),$$

corresponding to a map $T : (0, \infty) \to (0, \infty)$ given by

$$Tx = \frac{1}{2}\left(x + \frac{a}{x}\right). \tag{3.6}$$

Clearly, $x = \sqrt{a}$ is a fixed point of $T$. Moreover, given an approximation $x_n$ of $\sqrt{a}$, the average of $x_n$ and $a/x_n$ should be a better approximation provided that $x_n$ is not too small, so it is reasonable to expect that the sequence of approximations obtained by iteration of the fixed point equation converges to $\sqrt{a}$. We will prove that this is indeed true by finding an interval on which $T$ is a contraction with respect to the usual absolute value metric on $\mathbb{R}$.

First, let us see whether $T$ contracts at all. For $x_1, x_2 > 0$, we estimate that

$$\begin{aligned} |Tx_1 - Tx_2| &= \left|\frac{1}{2}\left(x_1 + \frac{a}{x_1}\right) - \frac{1}{2}\left(x_2 + \frac{a}{x_2}\right)\right| \\ &= \frac{1}{2}\left|1 - \frac{a}{x_1 x_2}\right| |x_1 - x_2|. \end{aligned}$$

It follows that $T$ contracts distances when $3x_1x_2 > a$. To satisfy this condition, we need to exclude arguments $x$ that are too small. Therefore, we consider the action of $T$ on an interval of the form $[b, \infty)$ with $b > 0$. This is a complete metric space because $[b, \infty)$ is a closed subset of $\mathbb{R}$ and $\mathbb{R}$ is complete.

In order to make a good choice for $b$ we first observe that

$$Tx = \sqrt{a} + \frac{(x - \sqrt{a})^2}{2x} \geq \sqrt{a} \tag{3.7}$$

for all $x > 0$. Therefore, the restriction of $T$ to $[\sqrt{a}, \infty)$ is well-defined, since

$$T\left([\sqrt{a}, \infty)\right) \subset [\sqrt{a}, \infty),$$

and $T$ is a contraction on $[\sqrt{a}, \infty)$ with $c = 1/2$. It follows that for any $x_0 \geq \sqrt{a}$, the sequence $x_n = T^n x_0$ converges to $\sqrt{a}$ as $n \to \infty$. Moreover, as shown in the proof of Theorem 3.2, the convergence is exponentially fast, with

$$\begin{aligned}|T^n x_0 - \sqrt{a}| &\leq |T^n x_0 - T^m x_0| \\ &\leq \frac{c^n}{1-c}|Tx_0 - x_0| \\ &\leq \frac{1}{2^{n-1}}\left|\frac{a}{x_0} - x_0\right|.\end{aligned}$$

If $0 < x_0 < \sqrt{a}$, then $x_1 > \sqrt{a}$, and subsequent iterates remain in $[\sqrt{a}, \infty)$, so the iterates converge for any starting guess $x_0 \in (0, \infty)$.

Newton's method for the solution of a nonlinear system of equations, discussed in Section 13.5, can also be formulated as a fixed point iteration.

### 3.3 Integral equations

A linear *Fredholm integral equation of the second kind* for an unknown function $f : [a, b] \to \mathbb{R}$ is an equation of the form

$$f(x) - \int_a^b k(x, y) f(y)\, dy = g(x), \tag{3.8}$$

where $k : [a, b] \times [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are given functions. A *Fredholm integral equation of the first kind* is an equation of the form

$$\int_a^b k(x, y) f(y)\, dy = g(x).$$

The integral equation (3.8) may be written as a fixed point equation $Tf = f$, where the map $T$ is defined by

$$Tf(x) = g(x) + \int_a^b k(x, y) f(y)\, dy. \tag{3.9}$$

**Theorem 3.3** Suppose that $k : [a, b] \times [a, b] \to \mathbb{R}$ is a continuous function such that

$$\sup_{a \leq x \leq b} \left\{ \int_a^b |k(x, y)|\, dy \right\} < 1, \tag{3.10}$$

and $g : [a, b] \to \mathbb{R}$ is a continuous function. Then there is a unique continuous function $f : [a, b] \to \mathbb{R}$ that satisfies (3.8).

***Proof.*** We prove this result by showing that, when (3.10) is satisfied, the map $T$ is a contraction on the normed space $C([a, b])$ with the uniform norm $\|\cdot\|_\infty$.

From Theorem 2.4, the space $C([a,b])$ is complete. Moreover, $T$ is a contraction since, for any $f_1, f_2 \in C([a,b])$, we have

$$\begin{aligned}
\|Tf_1 - Tf_2\|_\infty &= \sup_{a\le x\le b}\left|\int_a^b k(x,y)(f_1(y)-f_2(y))\,dy\right| \\
&\le \sup_{a\le x\le b}\int_a^b |k(x,y)|\,|f_1(y)-f_2(y)|\,dy \\
&\le \|f_1-f_2\|_\infty \sup_{a\le x\le b}\left\{\int_a^b |k(x,y)|\,dy\right\} \\
&\le c\|f_1-f_2\|_\infty,
\end{aligned}$$

where

$$c = \sup_{a\le x\le b}\left\{\int_a^b |k(x,y)|\,dy\right\} < 1.$$

The result then follows from the contraction mapping theorem. □

From the proof of the contraction mapping theorem, we can obtain the fixed point $f$ as a limit,

$$f = \lim_{n\to\infty} T^n f_0, \tag{3.11}$$

for any $f_0 \in C([a,b])$. It is interesting to reinterpret this limit as a series. We define a map $K : C([a,b]) \to C([a,b])$ by

$$Kf = \int_a^b k(x,y)f(y)\,dy.$$

The map $K$ is called a *Fredholm integral operator*, and the function $k$ is called the *kernel* of $K$. Equation (3.8) may be written as

$$(I-K)f = g, \tag{3.12}$$

where $I$ is the identity map, meaning that $If = f$. The contraction mapping $T$ is given by $Tf = g + Kf$, which implies that

$$\begin{aligned}
T^n f_0 &= g + K\left(g + \ldots + K\left(g + Kf_0\right)\right) \\
&= g + Kg + \ldots + K^n g + K^{n+1} f_0.
\end{aligned}$$

Using this equation in (3.11), we find that

$$f = \sum_{n=0}^{\infty} K^n g.$$

Since $f = (I - K)^{-1}g$, we may write this equation formally as

$$(I - K)^{-1} = \sum_{n=0}^{\infty} K^n. \tag{3.13}$$

This series is called the *Neumann series*. The use of the partial sums of this series to approximate the inverse is called the *Born approximation*. Explicitly, we have

$$\begin{aligned}&\left(I + K + K^2 + \ldots\right) f(x)\\ &\quad = f(x) + \int_a^b k(x,y)f(y)\,dy + \int_a^b \int_a^b k(x,y)k(y,z)f(z)\,dydz + \ldots.\end{aligned}$$

The Neumann series resembles the geometric series,

$$(1 - x)^{-1} = \sum_{n=0}^{\infty} x^n \qquad \text{for } |x| < 1.$$

In fact, (3.13) really is a geometric series that is absolutely convergent with respect to a suitable operator norm when $\|K\| < 1$ (see Exercise 5.17). This explains why we do not need a condition on $g$; equation (3.10) is a condition that ensures $I - K$ is invertible, and this only involves $k$.

## 3.4 Boundary value problems for differential equations

Consider a copper rod or pipe that is wrapped with imperfect insulation. We use the temperature outside the rod as the zero point of our temperature scale. We denote the spatial coordinate along the rod by $x$, nondimensionalized so that the length of the rod is one, and time by $t$. The temperature $u(x,t)$ of the rod then satisfies the following linear PDE,

$$u_t = u_{xx} - q(x)u, \tag{3.14}$$

where the subscripts denote partial derivatives,

$$u_t = \frac{\partial u}{\partial t}, \qquad u_{xx} = \frac{\partial^2 u}{\partial x^2}.$$

The lateral heat loss is proportional to the coefficient function $q(x)$ and the temperature difference $u$ between the rod and the outside. If the rod is perfectly insulated, then $q = 0$ and (3.14) is the one-dimensional *heat* or *diffusion equation*.

Equation (3.14) does not uniquely determine $u$, and it has to be supplemented by an initial condition

$$u(x,0) = u_0(x) \qquad \text{for } 0 < x < 1$$

that specifies the initial temperature $u_0(x)$ of the rod, and boundary conditions at the ends of the rod. We suppose that the ends $x = 0$ and $x = 1$ of the rod are kept at constant temperatures of $T_0$ and $T_1$, respectively. Then

$$u(0,t) = T_0, \qquad u(1,t) = T_1 \qquad \text{for all } t > 0.$$

As $t \to \infty$, the system "forgets" its initial state and approaches an equilibrium state $u = u(x)$, which satisfies the following boundary value problem (BVP) for an ODE:

$$-u''(x) + q(x)u(x) = 0, \qquad 0 < x < 1, \tag{3.15}$$

$$u(0) = T_0, \quad u(1) = T_1. \tag{3.16}$$

If $q$ is a constant function, then (3.15) is easy to solve explicitly; but if $q(x)$ is not constant, explicit integration is in general impossible. We will use the contraction mapping theorem to show that there is a unique solution of this BVP when $q$ is not too large.

First, we replace the nonhomogeneous boundary conditions (3.16) by the corresponding homogeneous conditions. To do this, we write $u$ as

$$u(x) = v(x) + u_p(x),$$

where $v$ is a new unknown function and $u_p$ is a function that satisfies the nonhomogeneous boundary conditions. A convenient choice for $u_p$ is the linear function

$$u_p(x) = T_0 + (T_1 - T_0)\, x. \tag{3.17}$$

The function $v$ then satisfies

$$-v'' + q(x)v(x) = f(x), \tag{3.18}$$

$$v(0) = 0, \quad v(1) = 0, \tag{3.19}$$

where $f = -qu_p$. The transfer of nonhomogeneous terms between the boundary conditions and the differential equation is a common procedure in the analysis of linear boundary value problems. The boundary value problem (3.18)–(3.19) is an example of a Sturm-Liouville problem, which we will study in Chapter 10. We will use the following proposition to reformulate this boundary value problem as a fixed point problem for a Fredholm integral operator.

**Proposition 3.4** Let $f : [0,1] \to \mathbb{R}$ be a continuous function. The unique solution $v$ of the boundary value problem

$$-v'' = f, \tag{3.20}$$

$$v(0) = 0, \quad v(1) = 0, \tag{3.21}$$

is given by

$$v(x) = \int_0^1 g(x,y) f(y)\, dy,$$

where

$$g(x,y) = \begin{cases} x(1-y) & \text{if } 0 \le x \le y \le 1, \\ y(1-x) & \text{if } 0 \le y \le x \le 1. \end{cases} \tag{3.22}$$

***Proof.*** Integrating (3.20) twice, we obtain that

$$v(x) = -\int_0^x \int_1^y f(s)\,ds\,dy + C_1 x + C_2,$$

where $C_1$ and $C_2$ are two real constants to be determined later. Integration by parts of the integral with respect to $y$ in this equation gives

$$\begin{aligned} v(x) &= -\left[ y \int_1^y f(s)\,ds \right]_0^x + \int_0^x y f(y)\,dy + C_1 x + C_2 \\ &= x \int_x^1 f(y)\,dy + \int_0^x y f(y)\,dy + C_1 x + C_2. \end{aligned}$$

We determine $C_1$ and $C_2$ from (3.21), which implies that

$$C_1 = -\int_0^1 y f(y)\,dy, \qquad C_2 = 0.$$

It follows that

$$v(x) = \int_0^x y(1-x) f(y)\,dy + \int_x^1 x(1-y) f(y)\,dy,$$

which is what we had to prove. □

The function $g(x,y)$ constructed in this proposition is called the *Green's function* of the differential operator $A = -d^2/dx^2$ in (3.20) with the Dirichlet boundary conditions (3.21). The inverse of the differential operator $A$ is an integral operator whose kernel is the Green's function. We will study Green's functions in greater detail in Chapter 10.

Replacing $f$ by $-qv + f$ in Proposition 3.4, we may rewrite (3.18) as an integral equation for $v$,

$$v(x) = -\int_0^1 g(x,y) q(y) v(y)\,dy + \int_0^1 g(x,y) f(y)\,dy.$$

This equation has the form

$$(I - K)v = h,$$

where the integral operator $K$ and the right-hand side $h$ are given by:

$$\begin{aligned} Kv(x) &= -\int_0^1 g(x,y) q(y) v(y)\,dy, \\ h(x) &= -\int_0^1 g(x,y) q(y) u_p(y)\,dy. \end{aligned} \tag{3.23}$$

Theorem 3.3 now implies the following result.

**Theorem 3.5** If $q : [0, 1] \to \mathbb{R}$ is continuous, and

$$\sup_{0 \le x \le 1} \left\{ \int_0^1 |g(x, y) q(y)| \, dy \right\} < 1, \tag{3.24}$$

where $g(x, y)$ is defined in (3.22), then the boundary value problem (3.15)–(3.16) has a unique solution.

Using (3.22), we find the estimate

$$\sup_{0 \le x \le 1} \left\{ \int_0^1 |g(x, y) q(y)| \, dy \right\} \le \frac{1}{8} \|q\|_\infty .$$

Thus, there is a unique solution of (3.15)–(3.16) for any continuous $q$ with $\|q\|_\infty < 8$. Existence or uniqueness may break down when $\|q\|_\infty$ is sufficiently large. For example, if $q = -n^2\pi^2$, where $n = 1, 2, 3, \ldots$, then the BVP

$$\begin{aligned} &-u'' - n^2\pi^2 u = 0, \\ &u(0) = 0, \qquad u(1) = 0, \end{aligned}$$

has the one-parameter family of solutions $u = c \sin n\pi x$, where $c$ is an arbitrary constant, and no solution may exist in the case of nonzero boundary conditions. Since $\pi^2 > 8$, this result is consistent with Theorem 3.5.

The nonuniqueness breaks down only if $q$ is negative. This is not physically realistic in the heat flow problem, where it would correspond to the flow of heat from cold to hot, but the same equation arises in many other problems with negative coefficient functions $q$.

If $q(x) > 0$ in $0 < x < 1$, we may prove uniqueness by a simple maximum principle argument. Suppose that $-u'' + q(x)u = 0$ and $u(0) = u(1) = 0$. Since $u$ is continuous, it attains its maximum and minimum on the interval $[0, 1]$. If $u$ attains its maximum at an interior point, then $u'' \le 0$ at that point, so $u = u''/q \le 0$. Since $u = 0$ at the endpoints, we conclude that

$$\max_{0 \le x \le 1} u(x) \le 0.$$

Similarly, at an interior minimum, we have $u'' \ge 0$, so $u \ge 0$, and therefore

$$\min_{0 \le x \le 1} u(x) \ge 0.$$

It follows that $u = 0$, so the solution of the boundary value problem is unique. Generalizations of this maximum principle argument apply to scalar elliptic partial differential equations, such as Laplace's equation.

### 3.5 Initial value problems for differential equations

The contraction mapping theorem may be used to prove the existence and uniqueness of solutions of the initial value problem for ordinary differential equations. We consider a first-order system of ODEs for a function $u(t)$ that takes values in $\mathbb{R}^n$,

$$\dot{u}(t) = f\left(t, u(t)\right), \tag{3.25}$$

$$u(t_0) = u_0. \tag{3.26}$$

The function $f(t,u)$ also takes values in $\mathbb{R}^n$, and is assumed to be a continuous function of $t$ and a Lipschitz continuous function of $u$ on a suitable domain. The initial value problem (3.25)–(3.26) can be reformulated as an integral equation,

$$u(t) = u_0 + \int_{t_0}^{t} f\left(s, u(s)\right)\, ds. \tag{3.27}$$

By the fundamental theorem of calculus, a continuous solution of (3.27) is a continuously differentiable solution of (3.25)–(3.26). Equation (3.27) may be written as a fixed point equation

$$u = Tu \tag{3.28}$$

for the map $T$ defined by

$$Tu(t) = u_0 + \int_{t_0}^{t} f(s, u(s))\, ds. \tag{3.29}$$

We want to find conditions which guarantee that $T$ is a contraction on a suitable space of continuous functions. The simplest such condition is given in the following definition.

**Definition 3.6** Suppose that $f : I \times \mathbb{R}^n \to \mathbb{R}^n$, where $I$ is an interval in $\mathbb{R}$. We say that $f(t,u)$ is a *globally Lipschitz continuous function of* $u$ *uniformly in* $t$ if there is a constant $C > 0$ such that

$$\|f(t,u) - f(t,v)\| \leq C\|u - v\| \qquad \text{for all } u, v \in \mathbb{R}^n \text{ and all } t \in I. \tag{3.30}$$

**Theorem 3.7** Suppose that $f : I \times \mathbb{R}^n \to \mathbb{R}^n$, where $I$ is an interval in $\mathbb{R}$ and $t_0$ is a point in the interior of $I$. If $f(t,u)$, is a continuous function of $(t,u)$ and a globally Lipschitz continuous function of $u$, uniformly in $t$, on $I \times \mathbb{R}^n$, then there is a unique continuously differentiable function $u : I \to \mathbb{R}^n$ that satisfies (3.25).

***Proof.*** We will show that $T$ is a contraction on the space of continuous functions defined on a time interval $t_0 \leq t \leq t_0 + \delta$, for sufficiently small $\delta$. Suppose that $u, v : [t_0, t_0 + \delta] \to \mathbb{R}^n$ are two continuous functions. Then, from (3.29) and (3.30), we estimate

$$
\begin{aligned}
\|Tu - Tv\|_\infty &= \sup_{t_0 \le t \le t_0+\delta} \|Tu(t) - Tv(t)\| \\
&= \sup_{t_0 \le t \le t_0+\delta} \left\| \int_{t_0}^{t} f(s, u(s)) - f(s, v(s))\, ds \right\| \\
&\le \sup_{t_0 \le t \le t_0+\delta} \int_{t_0}^{t} \|f(s, u(s)) - f(s, v(s))\|\, ds \\
&\le \sup_{t_0 \le t \le t_0+\delta} \int_{t_0}^{t} C\, \|u(s) - v(s)\|\, ds \\
&\le C\delta \|u - v\|_\infty .
\end{aligned}
$$

It follows that if $\delta < 1/C$, then $T$ is a contraction on $C([t_0, t_0 + \delta])$. Therefore, there is a unique solution $u : [t_0, t_0 + \delta] \to \mathbb{R}^n$. The argument holds for any $t_0 \in I$, and by covering $I$ with overlapping intervals of length less than $1/C$, we see that (3.25) has a unique continuous solution defined on all of $I$. The same proof applies for times $t_0 - \delta < t < t_0$. □

We may have $I = \mathbb{R}$ in this theorem, in which case the solution exists globally.

**Example 3.8** Linear ODEs with continuous coefficients have unique global solutions. Since higher order ODEs may be reduced to first-order systems, it is sufficient to consider a first-order linear system of the form

$$
\begin{aligned}
&\dot{u}(t) = A(t)u(t) + b(t), \\
&u(0) = u_0,
\end{aligned}
$$

where $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ is a continuous matrix-valued function (with respect to any matrix norm — see the discussion in Section 5.2 below) and $b : \mathbb{R} \to \mathbb{R}^n$ is a continuous vector-valued function. For any bounded interval $I \subset \mathbb{R}$, there exists a constant $C$ such that

$$
\|A(t)u\| \le C\|u\| \qquad \text{for all } t \in I \text{ and } u \in \mathbb{R}^n .
$$

Therefore,

$$
\|A(t)u - A(t)v\| \le C\|u - v\|,
$$

for all $u, v \in \mathbb{R}^n$ and $t \in I$, so the hypotheses of Theorem 3.7 are satisfied, and we have a unique continuous solution on $I$. Since $I$ is an arbitrary interval, we conclude that there is a unique continuous solution for all $t \in \mathbb{R}$.

The applications of Theorem 3.7 are not limited to linear ODEs.

**Example 3.9** Consider the nonlinear, scalar ODE given by

$$\begin{aligned} \dot{u}(t) &= \sqrt{a(t)^2 + u(t)^2}, \\ u(0) &= u_0, \end{aligned}$$

where $a : \mathbb{R} \to \mathbb{R}$ is a continuous function. Here,

$$f(t,u) = \sqrt{a(t)^2 + u^2}.$$

This function is globally Lipschitz, with

$$|f(t,u) - f(t,v)| \le |u - v| \qquad \text{for all } t, u, v, \in \mathbb{R},$$

since

$$\begin{aligned} \left| \sqrt{a^2+u^2} - \sqrt{a^2+v^2} \right| &= \frac{\left| (a^2+u^2) - (a^2+v^2) \right|}{\sqrt{a^2+u^2} + \sqrt{a^2+v^2}} \\ &\le |u-v| \frac{|u| + |v|}{\sqrt{a^2+u^2} + \sqrt{a^2+v^2}} \\ &\le |u-v|\,. \end{aligned}$$

Theorem 3.7 implies that there is a unique global solution of this ODE.

The global Lipschitz condition (3.30) plays two roles in Theorem 3.7. First, it ensures uniqueness, which may fail if $f$ is only a continuous function of $u$. Second, it implies that $f$ does not grow faster than a linear function of $u$ as $\|u\| \to \infty$. This is what guarantees global existence. If $f$ is a nonlinear function, such as $f(u) = u^2$ in Example 2.22, that satisfies a local Lipschitz condition but not a global Lipschitz condition, then "blow-up" may occur, so that the solution exists only locally.

The above proof may be modified to provide a local existence result.

**Theorem 3.10 (Local existence for ODEs)** Let $f : I \times \overline{B}_R(u_0) \to \mathbb{R}^n$, where

$$I = \{t \in \mathbb{R} \mid |t - t_0| \le T\}$$

is an interval in $\mathbb{R}$, and

$$\overline{B}_R(u_0) = \{u \in \mathbb{R}^n \mid \|u - u_0\| \le R\}$$

is the closed ball of radius $R > 0$ centered at $u_0 \in \mathbb{R}^n$. Suppose that $f(t,u)$ is continuous on $I \times \overline{B}_R(u_0)$ and Lipschitz continuous with respect to $u$ uniformly in $t$. Let

$$M = \sup\left\{ \|f(t,u)\| \mid t \in I \text{ and } u \in \overline{B}_R(u_0) \right\} < \infty.$$

Then the initial value problem

$$\dot{u} = f(u), \qquad u(t_0) = u_0$$

has a unique continuously differentiable local solution $u(t)$, defined in the time interval $|t-t_0|<\delta$, where $\delta = \min(T, R/M)$.

***Proof.*** We rewrite the initial value problem as a fixed point equation $u = Tu$, where

$$Tu(t) = u_0 + \int_{t_0}^{t} f\left(s, u(s)\right)\, ds.$$

For $0 < \eta < \delta$ we define

$$X = \left\{u : [t_0 - \eta, t_0 + \eta] \to \overline{B}_R(u_0) \mid u \text{ is continuous}\right\},$$

where $X$ is equipped with the sup-norm,

$$\|u\|_\infty = \sup_{|t-t_0|\leq\eta} \|u(t)\|.$$

We will show that $T$ maps $X$ into $X$, and is a contraction when $\eta$ is sufficiently small.

First, if $u \in X$, then

$$\|Tu(t) - u_0\| = \left\|\int_{t_0}^{t} f\left(s, u(s)\right)\, ds\right\| \leq M\eta < R.$$

Hence $Tu \in X$ so that $T : X \to X$.

Second, we estimate

$$\begin{aligned}\|Tu - Tv\|_\infty &= \sup_{|t-t_0|\leq\eta}\left\|\int_{t_0}^{t}\left[f\left(s,u(s)\right) - f\left(s,v(s)\right)\right] ds\right\| \\ &\leq C\eta\|u-v\|_\infty,\end{aligned}$$

where $C$ is a Lipschitz constant for $f$. Hence if we choose $\eta = 1/(2C)$ then $T$ is a contraction on $X$ and it has a unique fixed point.

Since $\eta$ depends only on the Lipschitz constant of $f$ and on the distance $R$ of the initial data from the boundary of $\overline{B}_R(u_0)$, repeated application of this result gives a unique local solution defined for $|t-t_0|<\delta$. □

A significant feature of this result is that if $f(t,u)$ is continuous for all $t \in \mathbb{R}$, then the existence time $\delta$ only depends on the norm of $u$. Thus, the only way in which the solution of an ODE can fail to exist, assuming that the vector field $f$ is Lipschitz continuous on any ball, is if $\|u(t)\|$ becomes unbounded. There are many functions that are bounded and continuous on an open interval which cannot be extended continuously to $\mathbb{R}$; for example, $u(t) = \sin(1/t)$ is bounded and continuous on $(-\infty, 0)$ but has no continuous extension to $(-\infty, 0]$. This kind of behavior cannot happen for solutions of ODEs with continuous right-hand sides, because the derivative of the solution cannot become large unless the solution itself becomes large. If we can prove that for every $T > 0$ any local solution satisfies an *a priori* estimate of the form

$$\|u(t)\| \leq R \qquad \text{for } |t| \leq T,$$

then the local existence theorem implies that the local solution can be extended to the interval $(-T, T)$, and hence to a global solution.

**Example 3.11** A *gradient flow* is defined by a system of ODEs of the form

$$\dot{u} = -\nabla V(u), \tag{3.31}$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is a smooth real-valued function of $u$, and $\nabla$ denotes the gradient with respect to $u$. The component form of this equation is

$$\dot{u}_i = -\frac{\partial V}{\partial u_i}.$$

Solutions of a gradient system flow "down hill" in the direction of decreasing $V$. It follows from (3.31) and the chain rule that

$$\dot{V}(u) = \nabla V(u) \cdot \dot{u} = -\|\nabla V(u)\|^2 \leq 0.$$

Thus, if $V_0 = V(u(0))$, we have

$$V(u(t)) \leq V_0 \qquad \text{for } t > 0$$

for any local solution. Therefore, if the set $\{u \in \mathbb{R}^n \mid V(u) \leq V_0\}$ is bounded (which is the case, for example, if $V(u) \to \infty$ as $\|u\| \to \infty$), then the solution of the initial value problem for (3.31) exists for all $t > 0$. The function $V$ is an example of a *Liapunov function*.

Most systems of ODEs cannot be written as a gradient system for any potential $V(u)$. If $f$ is a smooth vector field on $\mathbb{R}^n$, then $f = -\nabla V$ if and only if its components $f_i$ satisfy the integrability conditions that arise from the equality of mixed partial derivatives of $V$,

$$\frac{\partial f_i}{\partial u_j} = \frac{\partial f_j}{\partial u_i}.$$

With a suitable definition of the gradient of functionals on an infinite-dimensional space (see Chapter 13), a number of PDEs, such as the heat equation, can also be interpreted as gradient flows.

**Example 3.12** A *Hamiltonian system* of ODEs for $q(t), p(t) \in \mathbb{R}^n$ is a system of the form

$$\dot{q} = \nabla_p H, \qquad \dot{p} = -\nabla_q H, \tag{3.32}$$

where $H(q,p)$ is a given smooth function, $H : \mathbb{R}^{2n} \to \mathbb{R}$, called the *Hamiltonian*. The chain rule implies that

$$\dot{H} = \nabla_q H \cdot \dot{q} + \nabla_p H \cdot \dot{p} = 0,$$

so $H$ is constant on solutions. If the level sets of $H$ are bounded, then solutions of (3.32) exist globally in time.

## 3.6 References

General references for this chapter are Simmons [50] and Marsden and Hoffman [37]. For more about the logistic map and chaotic dynamical systems, see Devaney [8], Guckenheimer and Holmes [18], and Schroeder [51]. Hirsch and Smale [21] discusses gradient flows.

## 3.7 Exercises

**Exercise 3.1** Show that $T : \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = \frac{\pi}{2} + x - \tan^{-1} x$$

has no fixed point, and

$$|T(x) - T(y)| < |x - y| \qquad \text{for all } x \neq y \in \mathbb{R}.$$

Why doesn't this example contradict the contraction mapping theorem?

**Exercise 3.2** (a) Show that for any $y > 0$, the convergence of $T^n y$ to $\sqrt{x}$ is in fact faster than exponential. Start from (3.7) to obtain a good estimate for $|Ty - \sqrt{x}|$.

(b) Take $y = 1$ and $x = 2$. Find a good estimate of how large $n$ needs to be for $T^n 1$ and $\sqrt{2}$ to have identical first $d$ digits. For example, how many iterations of the map $T$ are sufficient to compute the first 63 digits of $\sqrt{2}$?

**Exercise 3.3** The *secant method* for solving an equation $f(x) = 0$, where $f : \mathbb{R} \to \mathbb{R}$, is a variant of Newton's method. Starting with two initial points $x_0$, $x_1$, we compute a sequence of iterates $(x_n)$ by:

$$x_{n+1} = x_n - f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \qquad \text{for } n = 1, 2, 3, \ldots$$

Formulate and prove a convergence theorem for the secant method with $f(x) = x^2 - 2$.

**Exercise 3.4** If $T$ satisfies (3.1) for $c < 1$, show that we can estimate the distance of the fixed point $x$ from the initial point $x_0$ of the fixed point iteration by

$$d(x, x_0) \leq \frac{1}{1-c} d(x_1, x_0).$$

**Exercise 3.5** An $n \times n$ matrix $A$ is said to be *diagonally dominant* if for each row the sum of the absolute values of the off-diagonal terms is less than the absolute value of the diagonal term. We write $A = D - L - U$ where $D$ is diagonal, $L$ is lower triangular, and $U$ is upper triangular. If $A$ is diagonally dominant, show that

$$\|L + U\|_\infty < \|D\|_\infty,$$

where the $\infty$-norm $\|\cdot\|_\infty$ of a matrix is defined in (5.9). Use the contraction mapping theorem to prove that if $A$ is diagonally dominant, then $A$ is invertible and that the following iteration schemes converge to a solution of the equation $Ax = b$:

$$x_{n+1} = D^{-1}(L + U)x_n + D^{-1}b, \tag{3.33}$$

$$x_{n+1} = (D - L)^{-1} U x_n + (D - L)^{-1} b. \tag{3.34}$$

What can you say about the rates of convergence?

Such iterative schemes provide an efficient way to compute numerical solutions of large, sparse linear systems. The iteration (3.33) is called *Jacobi's method*, and (3.34) is called the *Gauss-Seidel method.*

**Exercise 3.6** The following integral equation for $f : [-a, a] \to \mathbb{R}$ arises in a model of the motion of gas particles on a line:

$$f(x) = 1 + \frac{1}{\pi} \int_{-a}^{a} \frac{1}{1 + (x - y)^2} f(y)\, dy \qquad \text{for } -a \leq x \leq a.$$

Prove that this equation has a unique bounded, continuous solution for every $0 < a < \infty$. Prove that the solution is nonnegative. What can you say if $a = \infty$?

**Exercise 3.7** Prove that there is a unique solution of the following nonlinear BVP when the constant $\lambda$ is sufficiently small,

$$-u'' + \lambda \sin u = f(x),$$
$$u(0) = 0, \qquad u(1) = 0.$$

Here, $f : [0, 1] \to \mathbb{R}$ is a given continuous function. Write out the first few iterates of a uniformly convergent sequence of approximations, beginning with $u_0 = 0$.
Hint. Reformulate the problem as a nonlinear integral equation.