

The Riemann Integral

I know of some universities in England where the Lebesgue integral is *taught* in the first year of a mathematics degree instead of the Riemann integral, but I know of no universities in England where students *learn* the Lebesgue integral in the first year of a mathematics degree. (Approximate quotation attributed to T. W. Körner)

Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded (not necessarily continuous) function on a compact (closed, bounded) interval. We will define what it means for f to be Riemann integrable on $[a, b]$ and, in that case, define its Riemann integral $\int_a^b f$. The integral of f on $[a, b]$ is a real number whose geometrical interpretation is the signed area under the graph $y = f(x)$ for $a \leq x \leq b$. This number is also called the definite integral of f . By integrating f over an interval $[a, x]$ with varying right end-point, we get a function of x , called an indefinite integral of f .

The most important result about integration is the fundamental theorem of calculus, which states that integration and differentiation are inverse operations in an appropriately understood sense. Among other things, this connection enables us to compute many integrals explicitly. We will prove the fundamental theorem in the next chapter. In this chapter, we define the Riemann integral and prove some of its basic properties.

Integrability is a less restrictive condition on a function than differentiability. Generally speaking, integration makes functions smoother, while differentiation makes functions rougher. For example, the indefinite integral of every continuous function exists and is differentiable, whereas the derivative of a continuous function need not exist (and typically doesn't).

The Riemann integral is the simplest integral to define, and it allows one to integrate every continuous function as well as some not-too-badly discontinuous functions. There are, however, many other types of integrals, the most important of which is the Lebesgue integral. The Lebesgue integral allows one to integrate unbounded or highly discontinuous functions whose Riemann integrals do not exist,

and it has better mathematical properties than the Riemann integral. The definition of the Lebesgue integral is more involved, requiring the use of measure theory, and we will not discuss it here. In any event, the Riemann integral is adequate for many purposes, and even if one needs the Lebesgue integral, it is best to understand the Riemann integral first.

11.1. The supremum and infimum of functions

In this section we collect some results about the supremum and infimum of functions that we use to study Riemann integration. These results can be referred back to as needed.

From Definition 6.11, the supremum or infimum of a function is the supremum or infimum of its range, and results about the supremum or infimum of sets translate immediately to results about functions. There are, however, a few differences, which come from the fact that we often compare the values of functions at the same point, rather than all of their values simultaneously.

Inequalities and operations on functions are defined pointwise as usual; for example, if $f, g : A \rightarrow \mathbb{R}$, then $f \leq g$ means that $f(x) \leq g(x)$ for every $x \in A$, and $f + g : A \rightarrow \mathbb{R}$ is defined by $(f + g)(x) = f(x) + g(x)$.

Proposition 11.1. Suppose that $f, g : A \rightarrow \mathbb{R}$ and $f \leq g$. Then

$$\sup_A f \leq \sup_A g, \quad \inf_A f \leq \inf_A g.$$

Proof. If $\sup g = \infty$, then $\sup f \leq \sup g$. Otherwise, if $f \leq g$ and g is bounded from above, then

$$f(x) \leq g(x) \leq \sup_A g \quad \text{for every } x \in A.$$

Thus, f is bounded from above by $\sup_A g$, so $\sup_A f \leq \sup_A g$. Similarly, $-f \geq -g$ implies that $\sup_A(-f) \geq \sup_A(-g)$, so $\inf_A f \leq \inf_A g$. \square

Note that $f \leq g$ does not imply that $\sup_A f \leq \inf_A g$; to get that conclusion, we need to know that $f(x) \leq g(y)$ for all $x, y \in A$ and use Proposition 2.24.

Example 11.2. Define $f, g : [0, 1] \rightarrow \mathbb{R}$ by $f(x) = 2x$, $g(x) = 2x + 1$. Then $f < g$ and

$$\sup_{[0,1]} f = 2, \quad \inf_{[0,1]} f = 0, \quad \sup_{[0,1]} g = 3, \quad \inf_{[0,1]} g = 1.$$

Thus, $\sup f > \inf g$ even though $f < g$.

As for sets, the supremum and infimum of functions do not, in general, preserve strict inequalities, and a function need not attain its supremum or infimum even if it exists.

Example 11.3. Define $f : [0, 1] \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1, \\ 0 & \text{if } x = 1. \end{cases}$$

Then $f < 1$ on $[0, 1]$ but $\sup_{[0,1]} f = 1$, and there is no point $x \in [0, 1]$ such that $f(x) = 1$.

Next, we consider the supremum and infimum of linear combinations of functions. Multiplication of a function by a positive constant multiplies the inf or sup, while multiplication by a negative constant switches the inf and sup,

Proposition 11.4. Suppose that $f : A \rightarrow \mathbb{R}$ is a bounded function and $c \in \mathbb{R}$. If $c \geq 0$, then

$$\sup_A cf = c \sup_A f, \quad \inf_A cf = c \inf_A f.$$

If $c < 0$, then

$$\sup_A cf = c \inf_A f, \quad \inf_A cf = c \sup_A f.$$

Proof. Apply Proposition 2.23 to the set $\{cf(x) : x \in A\} = c\{f(x) : x \in A\}$. \square

For sums of functions, we get an inequality.

Proposition 11.5. If $f, g : A \rightarrow \mathbb{R}$ are bounded functions, then

$$\sup_A (f + g) \leq \sup_A f + \sup_A g, \quad \inf_A (f + g) \geq \inf_A f + \inf_A g.$$

Proof. Since $f(x) \leq \sup_A f$ and $g(x) \leq \sup_A g$ for every $x \in [a, b]$, we have

$$f(x) + g(x) \leq \sup_A f + \sup_A g.$$

Thus, $f + g$ is bounded from above by $\sup_A f + \sup_A g$, so

$$\sup_A (f + g) \leq \sup_A f + \sup_A g.$$

The proof for the infimum is analogous (or apply the result for the supremum to the functions $-f, -g$). \square

We may have strict inequality in Proposition 11.5 because f and g may take values close to their suprema (or infima) at different points.

Example 11.6. Define $f, g : [0, 1] \rightarrow \mathbb{R}$ by $f(x) = x$, $g(x) = 1 - x$. Then

$$\sup_{[0,1]} f = \sup_{[0,1]} g = \sup_{[0,1]} (f + g) = 1,$$

so $\sup(f + g) = 1$ but $\sup f + \sup g = 2$. Here, f attains its supremum at 1, while g attains its supremum at 0.

Finally, we prove some inequalities that involve the absolute value.

Proposition 11.7. If $f, g : A \rightarrow \mathbb{R}$ are bounded functions, then

$$\left| \sup_A f - \sup_A g \right| \leq \sup_A |f - g|, \quad \left| \inf_A f - \inf_A g \right| \leq \sup_A |f - g|.$$

Proof. Since $f = f - g + g$ and $f - g \leq |f - g|$, we get from Proposition 11.5 and Proposition 11.1 that

$$\sup_A f \leq \sup_A (f - g) + \sup_A g \leq \sup_A |f - g| + \sup_A g,$$

so

$$\sup_A f - \sup_A g \leq \sup_A |f - g|.$$

Exchanging f and g in this inequality, we get

$$\sup_A g - \sup_A f \leq \sup_A |f - g|,$$

which implies that

$$\left| \sup_A f - \sup_A g \right| \leq \sup_A |f - g|.$$

Replacing f by $-f$ and g by $-g$ in this inequality, we get

$$\left| \inf_A f - \inf_A g \right| \leq \sup_A |f - g|,$$

where we use the fact that $\sup(-f) = -\inf f$. □

Proposition 11.8. If $f, g : A \rightarrow \mathbb{R}$ are bounded functions such that

$$|f(x) - f(y)| \leq |g(x) - g(y)| \quad \text{for all } x, y \in A,$$

then

$$\sup_A f - \inf_A f \leq \sup_A g - \inf_A g.$$

Proof. The condition implies that for all $x, y \in A$, we have

$$f(x) - f(y) \leq |g(x) - g(y)| = \max[g(x), g(y)] - \min[g(x), g(y)] \leq \sup_A g - \inf_A g,$$

which implies that

$$\sup\{f(x) - f(y) : x, y \in A\} \leq \sup_A g - \inf_A g.$$

From Proposition 2.24, we have

$$\sup\{f(x) - f(y) : x, y \in A\} = \sup_A f - \inf_A f,$$

so the result follows. □

11.2. Definition of the integral

The definition of the integral is more involved than the definition of the derivative. The derivative is approximated by difference quotients, whereas the integral is approximated by upper and lower sums based on a partition of an interval.

We say that two intervals are almost disjoint if they are disjoint or intersect only at a common endpoint. For example, the intervals $[0, 1]$ and $[1, 3]$ are almost disjoint, whereas the intervals $[0, 2]$ and $[1, 3]$ are not.

Definition 11.9. Let I be a nonempty, compact interval. A partition of I is a finite collection $\{I_1, I_2, \dots, I_n\}$ of almost disjoint, nonempty, compact subintervals whose union is I .

A partition of $[a, b]$ with subintervals $I_k = [x_{k-1}, x_k]$ is determined by the set of endpoints of the intervals

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b.$$

Abusing notation, we will denote a partition P either by its intervals

$$P = \{I_1, I_2, \dots, I_n\}$$

or by the set of endpoints of the intervals

$$P = \{x_0, x_1, x_2, \dots, x_{n-1}, x_n\}.$$

We'll adopt either notation as convenient; the context should make it clear which one is being used. There is always one more endpoint than interval.

Example 11.10. The set of intervals

$$\{[0, 1/5], [1/5, 1/4], [1/4, 1/3], [1/3, 1/2], [1/2, 1]\}$$

is a partition of $[0, 1]$. The corresponding set of endpoints is

$$\{0, 1/5, 1/4, 1/3, 1/2, 1\}.$$

We denote the length of an interval $I = [a, b]$ by

$$|I| = b - a.$$

Note that the sum of the lengths $|I_k| = x_k - x_{k-1}$ of the almost disjoint subintervals in a partition $\{I_1, I_2, \dots, I_n\}$ of an interval I is equal to length of the whole interval. This is obvious geometrically; algebraically, it follows from the telescoping series

$$\begin{aligned} \sum_{k=1}^n |I_k| &= \sum_{k=1}^n (x_k - x_{k-1}) \\ &= x_n - x_{n-1} + x_{n-1} - x_{n-2} + \cdots + x_2 - x_1 + x_1 - x_0 \\ &= x_n - x_0 \\ &= |I|. \end{aligned}$$

Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is a bounded function on the compact interval $I = [a, b]$ with

$$M = \sup_I f, \quad m = \inf_I f.$$

If $P = \{I_1, I_2, \dots, I_n\}$ is a partition of I , let

$$M_k = \sup_{I_k} f, \quad m_k = \inf_{I_k} f.$$

These suprema and infima are well-defined, finite real numbers since f is bounded. Moreover,

$$m \leq m_k \leq M_k \leq M.$$

If f is continuous on the interval I , then it is bounded and attains its maximum and minimum values on each subinterval, but a bounded discontinuous function need not attain its supremum or infimum.

We define the upper Riemann sum of f with respect to the partition P by

$$U(f; P) = \sum_{k=1}^n M_k |I_k| = \sum_{k=1}^n M_k (x_k - x_{k-1}),$$

and the lower Riemann sum of f with respect to the partition P by

$$L(f; P) = \sum_{k=1}^n m_k |I_k| = \sum_{k=1}^n m_k (x_k - x_{k-1}).$$

Geometrically, $U(f; P)$ is the sum of the signed areas of rectangles based on the intervals I_k that lie above the graph of f , and $L(f; P)$ is the sum of the signed areas of rectangles that lie below the graph of f . Note that

$$m(b - a) \leq L(f; P) \leq U(f; P) \leq M(b - a).$$

Let $\Pi(a, b)$, or Π for short, denote the collection of all partitions of $[a, b]$. We define the upper Riemann integral of f on $[a, b]$ by

$$U(f) = \inf_{P \in \Pi} U(f; P).$$

The set $\{U(f; P) : P \in \Pi\}$ of all upper Riemann sums of f is bounded from below by $m(b - a)$, so this infimum is well-defined and finite. Similarly, the set $\{L(f; P) : P \in \Pi\}$ of all lower Riemann sums is bounded from above by $M(b - a)$, and we define the lower Riemann integral of f on $[a, b]$ by

$$L(f) = \sup_{P \in \Pi} L(f; P).$$

These upper and lower sums and integrals depend on the interval $[a, b]$ as well as the function f , but to simplify the notation we won't show this explicitly. A commonly used alternative notation for the upper and lower integrals is

$$U(f) = \int_a^{\bar{b}} f, \quad L(f) = \int_a^{\underline{b}} f.$$

Note the use of “lower-upper” and “upper-lower” approximations for the integrals: we take the infimum of the upper sums and the supremum of the lower sums. As we show in Proposition 11.22 below, we always have $L(f) \leq U(f)$, but in general the upper and lower integrals need not be equal. We define Riemann integrability by their equality.

Definition 11.11. A function $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable on $[a, b]$ if it is bounded and its upper integral $U(f)$ and lower integral $L(f)$ are equal. In that case, the Riemann integral of f on $[a, b]$, denoted by

$$\int_a^b f(x) dx, \quad \int_a^b f, \quad \int_{[a,b]} f$$

or similar notations, is the common value of $U(f)$ and $L(f)$.

An unbounded function is not Riemann integrable. In the following, “integrable” will mean “Riemann integrable, and “integral” will mean “Riemann integral” unless stated explicitly otherwise.

11.2.1. Examples. Let us illustrate the definition of Riemann integrability with a number of examples.

Example 11.12. Define $f : [0, 1] \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 1/x & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

Then

$$\int_0^1 \frac{1}{x} dx$$

isn't defined as a Riemann integral because f is unbounded. In fact, if

$$0 < x_1 < x_2 < \cdots < x_{n-1} < 1$$

is a partition of $[0, 1]$, then

$$\sup_{[0, x_1]} f = \infty,$$

so the upper Riemann sums of f are not well-defined.

An integral with an unbounded interval of integration, such as

$$\int_1^{\infty} \frac{1}{x} dx,$$

also isn't defined as a Riemann integral. In this case, a partition of $[1, \infty)$ into finitely many intervals contains at least one unbounded interval, so the corresponding Riemann sum is not well-defined. A partition of $[1, \infty)$ into bounded intervals (for example, $I_k = [k, k+1]$ with $k \in \mathbb{N}$) gives an infinite series rather than a finite Riemann sum, leading to questions of convergence.

One can interpret the integrals in this example as limits of Riemann integrals, or improper Riemann integrals,

$$\int_0^1 \frac{1}{x} dx = \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon}^1 \frac{1}{x} dx, \quad \int_1^{\infty} \frac{1}{x} dx = \lim_{r \rightarrow \infty} \int_1^r \frac{1}{x} dx,$$

but these are not proper Riemann integrals in the sense of Definition 11.11. Such improper Riemann integrals involve two limits — a limit of Riemann sums to define the Riemann integrals, followed by a limit of Riemann integrals. Both of the improper integrals in this example diverge to infinity. (See Section 12.4.)

Next, we consider some examples of bounded functions on compact intervals.

Example 11.13. The constant function $f(x) = 1$ on $[0, 1]$ is Riemann integrable, and

$$\int_0^1 1 dx = 1.$$

To show this, let $P = \{I_1, I_2, \dots, I_n\}$ be any partition of $[0, 1]$ with endpoints

$$\{0, x_1, x_2, \dots, x_{n-1}, 1\}.$$

Since f is constant,

$$M_k = \sup_{I_k} f = 1, \quad m_k = \inf_{I_k} f = 1 \quad \text{for } k = 1, \dots, n,$$

and therefore

$$U(f; P) = L(f; P) = \sum_{k=1}^n (x_k - x_{k-1}) = x_n - x_0 = 1.$$

Geometrically, this equation is the obvious fact that the sum of the areas of the rectangles over (or, equivalently, under) the graph of a constant function is exactly equal to the area under the graph. Thus, every upper and lower sum of f on $[0, 1]$ is equal to 1, which implies that the upper and lower integrals

$$U(f) = \inf_{P \in \Pi} U(f; P) = \inf\{1\} = 1, \quad L(f) = \sup_{P \in \Pi} L(f; P) = \sup\{1\} = 1$$

are equal, and the integral of f is 1.

More generally, the same argument shows that every constant function $f(x) = c$ is integrable and

$$\int_a^b c \, dx = c(b - a).$$

The following is an example of a discontinuous function that is Riemann integrable.

Example 11.14. The function

$$f(x) = \begin{cases} 0 & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x = 0 \end{cases}$$

is Riemann integrable, and

$$\int_0^1 f \, dx = 0.$$

To show this, let $P = \{I_1, I_2, \dots, I_n\}$ be a partition of $[0, 1]$. Then, since $f(x) = 0$ for $x > 0$,

$$M_k = \sup_{I_k} f = 0, \quad m_k = \inf_{I_k} f = 0 \quad \text{for } k = 2, \dots, n.$$

The first interval in the partition is $I_1 = [0, x_1]$, where $0 < x_1 \leq 1$, and

$$M_1 = 1, \quad m_1 = 0,$$

since $f(0) = 1$ and $f(x) = 0$ for $0 < x \leq x_1$. It follows that

$$U(f; P) = x_1, \quad L(f; P) = 0.$$

Thus, $L(f) = 0$ and

$$U(f) = \inf\{x_1 : 0 < x_1 \leq 1\} = 0,$$

so $U(f) = L(f) = 0$ are equal, and the integral of f is 0. In this example, the infimum of the upper Riemann sums is not attained and $U(f; P) > U(f)$ for every partition P .

A similar argument shows that a function $f : [a, b] \rightarrow \mathbb{R}$ that is zero except at finitely many points in $[a, b]$ is Riemann integrable with integral 0.

The next example is a bounded function on a compact interval whose Riemann integral doesn't exist.

Example 11.15. The Dirichlet function $f : [0, 1] \rightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, 1] \cap \mathbb{Q}, \\ 0 & \text{if } x \in [0, 1] \setminus \mathbb{Q}. \end{cases}$$

That is, f is one at every rational number and zero at every irrational number.

This function is not Riemann integrable. If $P = \{I_1, I_2, \dots, I_n\}$ is a partition of $[0, 1]$, then

$$M_k = \sup_{I_k} f = 1, \quad m_k = \inf_{I_k} f = 0,$$

since every interval of non-zero length contains both rational and irrational numbers. It follows that

$$U(f; P) = 1, \quad L(f; P) = 0$$

for every partition P of $[0, 1]$, so $U(f) = 1$ and $L(f) = 0$ are not equal.

The Dirichlet function is discontinuous at every point of $[0, 1]$, and the moral of the last example is that the Riemann integral of a highly discontinuous function need not exist. Nevertheless, some fairly discontinuous functions are still Riemann integrable.

Example 11.16. The Thomae function defined in Example 7.14 is Riemann integrable. The proof is left as an exercise.

Theorem 11.58 and Theorem 11.61 below give precise statements of the extent to which a Riemann integrable function can be discontinuous.

11.2.2. Refinements of partitions. As the previous examples illustrate, a direct verification of integrability from Definition 11.11 is unwieldy even for the simplest functions because we have to consider all possible partitions of the interval of integration. To give an effective analysis of Riemann integrability, we need to study how upper and lower sums behave under the refinement of partitions.

Definition 11.17. A partition $Q = \{J_1, J_2, \dots, J_m\}$ is a refinement of a partition $P = \{I_1, I_2, \dots, I_n\}$ if every interval I_k in P is an almost disjoint union of one or more intervals J_ℓ in Q .

Equivalently, if we represent partitions by their endpoints, then Q is a refinement of P if $Q \supset P$, meaning that every endpoint of P is an endpoint of Q . We don't require that every interval — or even any interval — in a partition has to be split into smaller intervals to obtain a refinement; for example, every partition is a refinement of itself.

Example 11.18. Consider the partitions of $[0, 1]$ with endpoints

$$P = \{0, 1/2, 1\}, \quad Q = \{0, 1/3, 2/3, 1\}, \quad R = \{0, 1/4, 1/2, 3/4, 1\}.$$

Thus, P , Q , and R partition $[0, 1]$ into intervals of equal length $1/2$, $1/3$, and $1/4$, respectively. Then Q is not a refinement of P but R is a refinement of P .

Given two partitions, neither one need be a refinement of the other. However, two partitions P , Q always have a common refinement; the smallest one is $R = P \cup Q$, meaning that the endpoints of R are exactly the endpoints of P or Q (or both).

Example 11.19. Let $P = \{0, 1/2, 1\}$ and $Q = \{0, 1/3, 2/3, 1\}$, as in Example 11.18. Then Q isn't a refinement of P and P isn't a refinement of Q . The partition $S = P \cup Q$, or

$$S = \{0, 1/3, 1/2, 2/3, 1\},$$

is a refinement of both P and Q . The partition S is not a refinement of R , but $T = R \cup S$, or

$$T = \{0, 1/4, 1/3, 1/2, 2/3, 3/4, 1\},$$

is a common refinement of all of the partitions $\{P, Q, R, S\}$.

As we show next, refining partitions decreases upper sums and increases lower sums. (The proof is easier to understand than it is to write out — draw a picture!)

Theorem 11.20. Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is bounded, P is a partitions of $[a, b]$, and Q is refinement of P . Then

$$U(f; Q) \leq U(f; P), \quad L(f; P) \leq L(f; Q).$$

Proof. Let

$$P = \{I_1, I_2, \dots, I_n\}, \quad Q = \{J_1, J_2, \dots, J_m\}$$

be partitions of $[a, b]$, where Q is a refinement of P , so $m \geq n$. We list the intervals in increasing order of their endpoints. Define

$$M_k = \sup_{I_k} f, \quad m_k = \inf_{I_k} f, \quad M'_\ell = \sup_{J_\ell} f, \quad m'_\ell = \inf_{J_\ell} f.$$

Since Q is a refinement of P , each interval I_k in P is an almost disjoint union of intervals in Q , which we can write as

$$I_k = \bigcup_{\ell=p_k}^{q_k} J_\ell$$

for some indices $p_k \leq q_k$. If $p_k < q_k$, then I_k is split into two or more smaller intervals in Q , and if $p_k = q_k$, then I_k belongs to both P and Q . Since the intervals are listed in order, we have

$$p_1 = 1, \quad p_{k+1} = q_k + 1, \quad q_n = m.$$

If $p_k \leq \ell \leq q_k$, then $J_\ell \subset I_k$, so

$$M'_\ell \leq M_k, \quad m_k \geq m'_\ell \quad \text{for } p_k \leq \ell \leq q_k.$$

Using the fact that the sum of the lengths of the J -intervals is the length of the corresponding I -interval, we get that

$$\sum_{\ell=p_k}^{q_k} M'_\ell |J_\ell| \leq \sum_{\ell=p_k}^{q_k} M_k |J_\ell| = M_k \sum_{\ell=p_k}^{q_k} |J_\ell| = M_k |I_k|.$$

It follows that

$$U(f; Q) = \sum_{\ell=1}^m M'_\ell |J_\ell| = \sum_{k=1}^n \sum_{\ell=p_k}^{q_k} M'_\ell |J_\ell| \leq \sum_{k=1}^n M_k |I_k| = U(f; P).$$

Similarly,

$$\sum_{\ell=p_k}^{q_k} m'_\ell |J_\ell| \geq \sum_{\ell=p_k}^{q_k} m_k |J_\ell| = m_k |I_k|,$$

and

$$L(f; Q) = \sum_{k=1}^n \sum_{\ell=p_k}^{q_k} m'_\ell |J_\ell| \geq \sum_{k=1}^n m_k |I_k| = L(f; P),$$

which proves the result. \square

It follows from this theorem that all lower sums are less than or equal to all upper sums, not just the lower and upper sums associated with the same partition.

Proposition 11.21. If $f : [a, b] \rightarrow \mathbb{R}$ is bounded and P, Q are partitions of $[a, b]$, then

$$L(f; P) \leq U(f; Q).$$

Proof. Let R be a common refinement of P and Q . Then, by Theorem 11.20,

$$L(f; P) \leq L(f; R), \quad U(f; R) \leq U(f; Q).$$

It follows that

$$L(f; P) \leq L(f; R) \leq U(f; R) \leq U(f; Q).$$

□

An immediate consequence of this result is that the lower integral is always less than or equal to the upper integral.

Proposition 11.22. If $f : [a, b] \rightarrow \mathbb{R}$ is bounded, then

$$L(f) \leq U(f).$$

Proof. Let

$$A = \{L(f; P) : P \in \Pi\}, \quad B = \{U(f; P) : P \in \Pi\}.$$

From Proposition 11.21, $L \leq U$ for every $L \in A$ and $U \in B$, so Proposition 2.22 implies that $\sup A \leq \inf B$, or $L(f) \leq U(f)$. □

11.3. The Cauchy criterion for integrability

The following theorem gives a criterion for integrability that is analogous to the Cauchy condition for the convergence of a sequence.

Theorem 11.23. A bounded function $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable if and only if for every $\epsilon > 0$ there exists a partition P of $[a, b]$, which may depend on ϵ , such that

$$U(f; P) - L(f; P) < \epsilon.$$

Proof. First, suppose that the condition holds. Let $\epsilon > 0$ and choose a partition P that satisfies the condition. Then, since $U(f) \leq U(f; P)$ and $L(f; P) \leq L(f)$, we have

$$0 \leq U(f) - L(f) \leq U(f; P) - L(f; P) < \epsilon.$$

Since this inequality holds for every $\epsilon > 0$, we must have $U(f) - L(f) = 0$, and f is integrable.

Conversely, suppose that f is integrable. Given any $\epsilon > 0$, there are partitions Q, R such that

$$U(f; Q) < U(f) + \frac{\epsilon}{2}, \quad L(f; R) > L(f) - \frac{\epsilon}{2}.$$

Let P be a common refinement of Q and R . Then, by Theorem 11.20,

$$U(f; P) - L(f; P) \leq U(f; Q) - L(f; R) < U(f) - L(f) + \epsilon.$$

Since $U(f) = L(f)$, the condition follows. □

If $U(f; P) - L(f; P) < \epsilon$, then $U(f; Q) - L(f; Q) < \epsilon$ for every refinement Q of P , so the Cauchy condition means that a function is integrable if and only if its upper and lower sums get arbitrarily close together for all sufficiently refined partitions.

It is worth considering in more detail what the Cauchy condition in Theorem 11.23 implies about the behavior of a Riemann integrable function.

Definition 11.24. The oscillation of a bounded function f on a set A is

$$\operatorname{osc}_A f = \sup_A f - \inf_A f.$$

If $f : [a, b] \rightarrow \mathbb{R}$ is bounded and $P = \{I_1, I_2, \dots, I_n\}$ is a partition of $[a, b]$, then

$$U(f; P) - L(f; P) = \sum_{k=1}^n \sup_{I_k} f \cdot |I_k| - \sum_{k=1}^n \inf_{I_k} f \cdot |I_k| = \sum_{k=1}^n \operatorname{osc}_{I_k} f \cdot |I_k|.$$

A function f is Riemann integrable if we can make $U(f; P) - L(f; P)$ as small as we wish. This is the case if we can find a sufficiently refined partition P such that the oscillation of f on most intervals is arbitrarily small, and the sum of the lengths of the remaining intervals (where the oscillation of f is large) is arbitrarily small. For example, the discontinuous function in Example 11.14 has zero oscillation on every interval except the first one, where the function has oscillation one, but the length of that interval can be made as small as we wish.

Thus, roughly speaking, a function is Riemann integrable if it oscillates by an arbitrary small amount except on a finite collection of intervals whose total length is arbitrarily small. Theorem 11.58 gives a precise statement.

One direct consequence of the Cauchy criterion is that a function is integrable if we can estimate its oscillation by the oscillation of an integrable function.

Proposition 11.25. Suppose that $f, g : [a, b] \rightarrow \mathbb{R}$ are bounded functions and g is integrable on $[a, b]$. If there exists a constant $C \geq 0$ such that

$$\operatorname{osc}_I f \leq C \operatorname{osc}_I g$$

on every interval $I \subset [a, b]$, then f is integrable.

Proof. If $P = \{I_1, I_2, \dots, I_n\}$ is a partition of $[a, b]$, then

$$\begin{aligned} U(f; P) - L(f; P) &= \sum_{k=1}^n \operatorname{osc}_{I_k} f \cdot |I_k| \\ &\leq C \sum_{k=1}^n \operatorname{osc}_{I_k} g \cdot |I_k| \\ &\leq C [U(g; P) - L(g; P)]. \end{aligned}$$

Thus, f satisfies the Cauchy criterion in Theorem 11.23 if g does, which proves that f is integrable if g is integrable. \square

We can also use the Cauchy criterion to give a sequential characterization of integrability.

Theorem 11.26. A bounded function $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable if and only if there is a sequence (P_n) of partitions such that

$$\lim_{n \rightarrow \infty} [U(f; P_n) - L(f; P_n)] = 0.$$

In that case,

$$\int_a^b f = \lim_{n \rightarrow \infty} U(f; P_n) = \lim_{n \rightarrow \infty} L(f; P_n).$$

Proof. First, suppose that the condition holds. Then, given $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that $U(f; P_n) - L(f; P_n) < \epsilon$, so Theorem 11.23 implies that f is integrable and $U(f) = L(f)$.

Furthermore, since $U(f) \leq U(f; P_n)$ and $L(f; P_n) \leq L(f)$, we have

$$0 \leq U(f; P_n) - U(f) = U(f; P_n) - L(f) \leq U(f; P_n) - L(f; P_n).$$

Since the limit of the right-hand side is zero, the ‘squeeze’ theorem implies that

$$\lim_{n \rightarrow \infty} U(f; P_n) = U(f) = \int_a^b f.$$

It also follows that

$$\lim_{n \rightarrow \infty} L(f; P_n) = \lim_{n \rightarrow \infty} U(f; P_n) - \lim_{n \rightarrow \infty} [U(f; P_n) - L(f; P_n)] = \int_a^b f.$$

Conversely, if f is integrable then, by Theorem 11.23, for every $n \in \mathbb{N}$ there exists a partition P_n such that

$$0 \leq U(f; P_n) - L(f; P_n) < \frac{1}{n},$$

and $U(f; P_n) - L(f; P_n) \rightarrow 0$ as $n \rightarrow \infty$. □

Note that if the limits of $U(f; P_n)$ and $L(f; P_n)$ both exist and are equal, then

$$\lim_{n \rightarrow \infty} [U(f; P_n) - L(f; P_n)] = \lim_{n \rightarrow \infty} U(f; P_n) - \lim_{n \rightarrow \infty} L(f; P_n),$$

so the conditions of the theorem are satisfied. Conversely, the proof of the theorem shows that if the limit of $U(f; P_n) - L(f; P_n)$ is zero, then the limits of $U(f; P_n)$ and $L(f; P_n)$ both exist and are equal. This isn’t true for general sequences, where one may have $\lim(a_n - b_n) = 0$ even though $\lim a_n$ and $\lim b_n$ don’t exist.

Theorem 11.26 provides one way to prove the existence of an integral and, in some cases, evaluate it.

Example 11.27. Consider the function $f(x) = x^2$ on $[0, 1]$. Let P_n be the partition of $[0, 1]$ into n -intervals of equal length $1/n$ with endpoints $x_k = k/n$ for $k = 0, 1, 2, \dots, n$. If $I_k = [(k-1)/n, k/n]$ is the k th interval, then

$$\sup_{I_k} f = x_k^2, \quad \inf_{I_k} f = x_{k-1}^2$$

since f is increasing. Using the formula for the sum of squares

$$\sum_{k=1}^n k^2 = \frac{1}{6}n(n+1)(2n+1),$$

we get

$$U(f; P_n) = \sum_{k=1}^n x_k^2 \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^n k^2 = \frac{1}{6} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right)$$

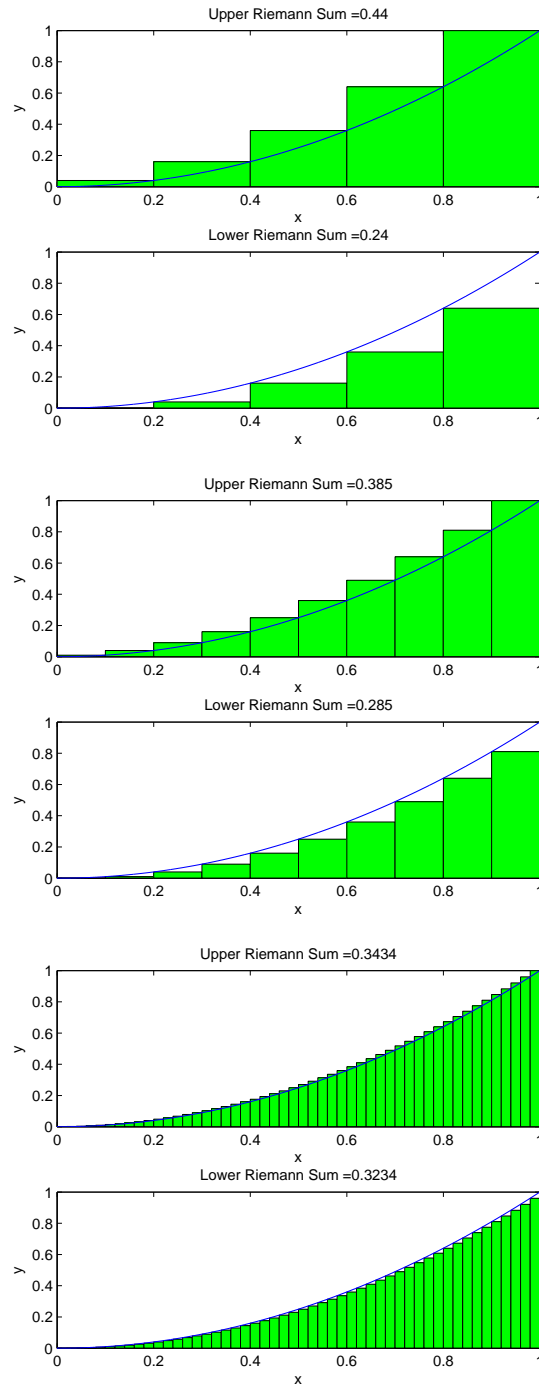


Figure 1. Upper and lower Riemann sums for Example 11.27 with $n = 5, 10, 50$ subintervals of equal length.

and

$$L(f; P_n) = \sum_{k=1}^n x_{k-1}^2 \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n-1} k^2 = \frac{1}{6} \left(1 - \frac{1}{n}\right) \left(2 - \frac{1}{n}\right).$$

(See Figure 11.27.) It follows that

$$\lim_{n \rightarrow \infty} U(f; P_n) = \lim_{n \rightarrow \infty} L(f; P_n) = \frac{1}{3},$$

and Theorem 11.26 implies that x^2 is integrable on $[0, 1]$ with

$$\int_0^1 x^2 dx = \frac{1}{3}.$$

The fundamental theorem of calculus, Theorem 12.1 below, provides a much easier way to evaluate this integral, but the Riemann sums provide the basic definition of the integral.

11.4. Continuous and monotonic functions

The Cauchy criterion leads to the following fundamental result that every continuous function is Riemann integrable. To prove this result, we use the fact that a continuous function oscillates by an arbitrarily small amount on every interval of a sufficiently refined partition.

Theorem 11.28. A continuous function $f : [a, b] \rightarrow \mathbb{R}$ on a compact interval is Riemann integrable.

Proof. A continuous function on a compact set is bounded, so we just need to verify the Cauchy condition in Theorem 11.23.

Let $\epsilon > 0$. A continuous function on a compact set is uniformly continuous, so there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\epsilon}{b-a} \quad \text{for all } x, y \in [a, b] \text{ such that } |x - y| < \delta.$$

Choose a partition $P = \{I_1, I_2, \dots, I_n\}$ of $[a, b]$ such that $|I_k| < \delta$ for every k ; for example, we can take n intervals of equal length $(b-a)/n$ with $n > (b-a)/\delta$.

Since f is continuous, it attains its maximum and minimum values M_k and m_k on the compact interval I_k at points x_k and y_k in I_k . These points satisfy $|x_k - y_k| < \delta$, so

$$M_k - m_k = f(x_k) - f(y_k) < \frac{\epsilon}{b-a}.$$

The upper and lower sums of f therefore satisfy

$$\begin{aligned} U(f; P) - L(f; P) &= \sum_{k=1}^n M_k |I_k| - \sum_{k=1}^n m_k |I_k| \\ &= \sum_{k=1}^n (M_k - m_k) |I_k| \\ &< \frac{\epsilon}{b-a} \sum_{k=1}^n |I_k| \\ &< \epsilon, \end{aligned}$$

and Theorem 11.23 implies that f is integrable. \square

Example 11.29. The function $f(x) = x^2$ on $[0, 1]$ considered in Example 11.27 is integrable since it is continuous.

Another class of integrable functions consists of monotonic (increasing or decreasing) functions.

Theorem 11.30. A monotonic function $f : [a, b] \rightarrow \mathbb{R}$ on a compact interval is Riemann integrable.

Proof. Suppose that f is monotonic increasing, meaning that $f(x) \leq f(y)$ for $x \leq y$. Let $P_n = \{I_1, I_2, \dots, I_n\}$ be a partition of $[a, b]$ into n intervals $I_k = [x_{k-1}, x_k]$, of equal length $(b-a)/n$, with endpoints

$$x_k = a + (b-a)\frac{k}{n}, \quad k = 0, 1, \dots, n.$$

Since f is increasing,

$$M_k = \sup_{I_k} f = f(x_k), \quad m_k = \inf_{I_k} f = f(x_{k-1}).$$

Hence, summing a telescoping series, we get

$$\begin{aligned} U(f; P_n) - L(f; P_n) &= \sum_{k=1}^n (M_k - m_k) (x_k - x_{k-1}) \\ &= \frac{b-a}{n} \sum_{k=1}^n [f(x_k) - f(x_{k-1})] \\ &= \frac{b-a}{n} [f(b) - f(a)]. \end{aligned}$$

It follows that $U(f; P_n) - L(f; P_n) \rightarrow 0$ as $n \rightarrow \infty$, and Theorem 11.26 implies that f is integrable.

The proof for a monotonic decreasing function f is similar, with

$$\sup_{I_k} f = f(x_{k-1}), \quad \inf_{I_k} f = f(x_k),$$

or we can apply the result for increasing functions to $-f$ and use Theorem 11.32 below. \square

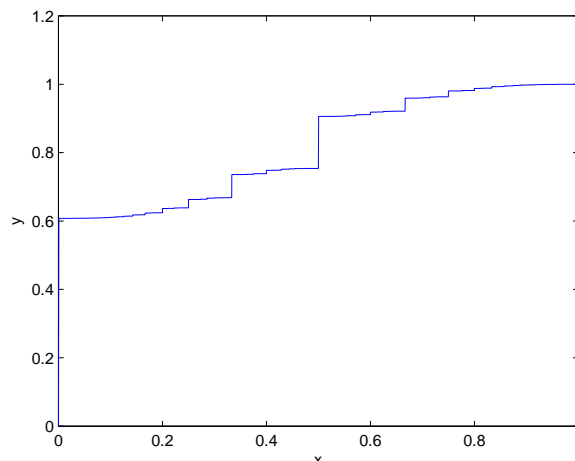


Figure 2. The graph of the monotonic function in Example 11.31 with a countably infinite, dense set of jump discontinuities.

Monotonic functions needn't be continuous, and they may be discontinuous at a countably infinite number of points.

Example 11.31. Let $\{q_k : k \in \mathbb{N}\}$ be an enumeration of the rational numbers in $[0, 1)$ and let (a_k) be a sequence of strictly positive real numbers such that

$$\sum_{k=1}^{\infty} a_k = 1.$$

Define $f : [0, 1] \rightarrow \mathbb{R}$ by

$$f(x) = \sum_{k \in Q(x)} a_k, \quad Q(x) = \{k \in \mathbb{N} : q_k \in [0, x)\}.$$

for $x > 0$, and $f(0) = 0$. That is, $f(x)$ is obtained by summing the terms in the series whose indices k correspond to the rational numbers such that $0 \leq q_k < x$.

For $x = 1$, this sum includes all the terms in the series, so $f(1) = 1$. For every $0 < x < 1$, there are infinitely many terms in the sum, since the rationals are dense in $[0, x)$, and f is increasing, since the number of terms increases with x . By Theorem 11.30, f is Riemann integrable on $[0, 1]$. Although f is integrable, it has a countably infinite number of jump discontinuities at every rational number in $[0, 1)$, which are dense in $[0, 1]$. The function is continuous elsewhere (the proof is left as an exercise).

Figure 2 shows the graph of f corresponding to the enumeration

$$\{0, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 1/6, 5/6, 1/7, \dots\}$$

of the rational numbers in $[0, 1)$ and

$$a_k = \frac{6}{\pi^2} \frac{1}{k^2}.$$

What is its Riemann integral?

11.5. Linearity, monotonicity, and additivity

The integral has the following three basic properties.

(1) Linearity:

$$\int_a^b cf = c \int_a^b f, \quad \int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

(2) Monotonicity: if $f \leq g$, then

$$\int_a^b f \leq \int_a^b g.$$

(3) Additivity: if $a < c < b$, then

$$\int_a^c f + \int_c^b f = \int_a^b f.$$

These properties are analogous to the corresponding properties of sums (or convergent series):

$$\begin{aligned} \sum_{k=1}^n ca_k &= c \sum_{k=1}^n a_k, & \sum_{k=1}^n (a_k + b_k) &= \sum_{k=1}^n a_k + \sum_{k=1}^n b_k; \\ \sum_{k=1}^n a_k &\leq \sum_{k=1}^n b_k & \text{if } a_k &\leq b_k; \\ \sum_{k=1}^m a_k + \sum_{k=m+1}^n a_k &= \sum_{k=1}^n a_k. \end{aligned}$$

In this section, we prove these properties and derive a few of their consequences.

11.5.1. Linearity. We begin by proving the linearity. First we prove linearity with respect to scalar multiplication and then linearity with respect to sums.

Theorem 11.32. If $f : [a, b] \rightarrow \mathbb{R}$ is integrable and $c \in \mathbb{R}$, then cf is integrable and

$$\int_a^b cf = c \int_a^b f.$$

Proof. Suppose that $c \geq 0$. Then for any set $A \subset [a, b]$, we have

$$\sup_A cf = c \sup_A f, \quad \inf_A cf = c \inf_A f,$$

so $U(cf; P) = cU(f; P)$ for every partition P . Taking the infimum over the set Π of all partitions of $[a, b]$, we get

$$U(cf) = \inf_{P \in \Pi} U(cf; P) = \inf_{P \in \Pi} cU(f; P) = c \inf_{P \in \Pi} U(f; P) = cU(f).$$

Similarly, $L(cf; P) = cL(f; P)$ and $L(cf) = cL(f)$. If f is integrable, then

$$U(cf) = cU(f) = cL(f) = L(cf),$$

which shows that cf is integrable and

$$\int_a^b cf = c \int_a^b f.$$

Now consider $-f$. Since

$$\sup_A(-f) = -\inf_A f, \quad \inf_A(-f) = -\sup_A f,$$

we have

$$U(-f; P) = -L(f; P), \quad L(-f; P) = -U(f; P).$$

Therefore

$$\begin{aligned} U(-f) &= \inf_{P \in \Pi} U(-f; P) = \inf_{P \in \Pi} [-L(f; P)] = -\sup_{P \in \Pi} L(f; P) = -L(f), \\ L(-f) &= \sup_{P \in \Pi} L(-f; P) = \sup_{P \in \Pi} [-U(f; P)] = -\inf_{P \in \Pi} U(f; P) = -U(f). \end{aligned}$$

Hence, $-f$ is integrable if f is integrable and

$$\int_a^b (-f) = -\int_a^b f.$$

Finally, if $c < 0$, then $c = -|c|$, and a successive application of the previous results shows that cf is integrable with $\int_a^b cf = c \int_a^b f$. \square

Next, we prove the linearity of the integral with respect to sums. If f, g are bounded, then $f + g$ is bounded and

$$\sup_I(f + g) \leq \sup_I f + \sup_I g, \quad \inf_I(f + g) \geq \inf_I f + \inf_I g.$$

It follows that

$$\operatorname{osc}_I(f + g) \leq \operatorname{osc}_I f + \operatorname{osc}_I g,$$

so $f + g$ is integrable if f, g are integrable. In general, however, the upper (or lower) sum of $f + g$ needn't be the sum of the corresponding upper (or lower) sums of f and g . As a result, we don't get

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g$$

simply by adding upper and lower sums. Instead, we prove this equality by estimating the upper and lower integrals of $f + g$ from above and below by those of f and g .

Theorem 11.33. If $f, g : [a, b] \rightarrow \mathbb{R}$ are integrable functions, then $f + g$ is integrable, and

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

Proof. We first prove that if $f, g : [a, b] \rightarrow \mathbb{R}$ are bounded, but not necessarily integrable, then

$$U(f + g) \leq U(f) + U(g), \quad L(f + g) \geq L(f) + L(g).$$

Suppose that $P = \{I_1, I_2, \dots, I_n\}$ is a partition of $[a, b]$. Then

$$\begin{aligned} U(f+g; P) &= \sum_{k=1}^n \sup_{I_k} (f+g) \cdot |I_k| \\ &\leq \sum_{k=1}^n \sup_{I_k} f \cdot |I_k| + \sum_{k=1}^n \sup_{I_k} g \cdot |I_k| \\ &\leq U(f; P) + U(g; P). \end{aligned}$$

Let $\epsilon > 0$. Since the upper integral is the infimum of the upper sums, there are partitions Q, R such that

$$U(f; Q) < U(f) + \frac{\epsilon}{2}, \quad U(g; R) < U(g) + \frac{\epsilon}{2},$$

and if P is a common refinement of Q and R , then

$$U(f; P) < U(f) + \frac{\epsilon}{2}, \quad U(g; P) < U(g) + \frac{\epsilon}{2}.$$

It follows that

$$U(f+g) \leq U(f+g; P) \leq U(f; P) + U(g; P) < U(f) + U(g) + \epsilon.$$

Since this inequality holds for arbitrary $\epsilon > 0$, we must have $U(f+g) \leq U(f) + U(g)$.

Similarly, we have $L(f+g; P) \geq L(f; P) + L(g; P)$ for all partitions P , and for every $\epsilon > 0$, we get $L(f+g) > L(f) + L(g) - \epsilon$, so $L(f+g) \geq L(f) + L(g)$.

For integrable functions f and g , it follows that

$$U(f+g) \leq U(f) + U(g) = L(f) + L(g) \leq L(f+g).$$

Since $U(f+g) \geq L(f+g)$, we have $U(f+g) = L(f+g)$ and $f+g$ is integrable. Moreover, there is equality throughout the previous inequality, which proves the result. \square

Although the integral is linear, the upper and lower integrals of non-integrable functions are not, in general, linear.

Example 11.34. Define $f, g : [0, 1] \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, 1] \cap \mathbb{Q}, \\ 0 & \text{if } x \in [0, 1] \setminus \mathbb{Q}, \end{cases} \quad g(x) = \begin{cases} 0 & \text{if } x \in [0, 1] \cap \mathbb{Q}, \\ 1 & \text{if } x \in [0, 1] \setminus \mathbb{Q}. \end{cases}$$

That is, f is the Dirichlet function and $g = 1 - f$. Then

$$U(f) = U(g) = 1, \quad L(f) = L(g) = 0, \quad U(f+g) = L(f+g) = 1,$$

so

$$U(f+g) < U(f) + U(g), \quad L(f+g) > L(f) + L(g).$$

The product of integrable functions is also integrable, as is the quotient provided it remains bounded. Unlike the integral of the sum, however, there is no way to express the integral of the product $\int fg$ in terms of $\int f$ and $\int g$.

Theorem 11.35. If $f, g : [a, b] \rightarrow \mathbb{R}$ are integrable, then $fg : [a, b] \rightarrow \mathbb{R}$ is integrable. If, in addition, $g \neq 0$ and $1/g$ is bounded, then $f/g : [a, b] \rightarrow \mathbb{R}$ is integrable.

Proof. First, we show that the square of an integrable function is integrable. If f is integrable, then f is bounded, with $|f| \leq M$ for some $M \geq 0$. For all $x, y \in [a, b]$, we have

$$|f^2(x) - f^2(y)| = |f(x) + f(y)| \cdot |f(x) - f(y)| \leq 2M|f(x) - f(y)|.$$

Taking the supremum of this inequality over $x, y \in I \subset [a, b]$ and using Proposition 11.8, we get that

$$\sup_I(f^2) - \inf_I(f^2) \leq 2M \left[\sup_I f - \inf_I f \right],$$

meaning that

$$\operatorname{osc}_I(f^2) \leq 2M \operatorname{osc}_I f.$$

It follows from Proposition 11.25 that f^2 is integrable if f is integrable.

Since the integral is linear, we then see from the identity

$$fg = \frac{1}{4} [(f+g)^2 - (f-g)^2]$$

that fg is integrable if f, g are integrable. We remark that the trick of representing a product as a difference of squares isn't a new one: the ancient Babylonian apparently used this identity, together with a table of squares, to compute products.

In a similar way, if $g \neq 0$ and $|1/g| \leq M$, then

$$\left| \frac{1}{g(x)} - \frac{1}{g(y)} \right| = \frac{|g(x) - g(y)|}{|g(x)g(y)|} \leq M^2 |g(x) - g(y)|.$$

Taking the supremum of this equation over $x, y \in I \subset [a, b]$, we get

$$\sup_I \left(\frac{1}{g} \right) - \inf_I \left(\frac{1}{g} \right) \leq M^2 \left[\sup_I g - \inf_I g \right],$$

meaning that $\operatorname{osc}_I(1/g) \leq M^2 \operatorname{osc}_I g$, and Proposition 11.25 implies that $1/g$ is integrable if g is integrable. Therefore $f/g = f \cdot (1/g)$ is integrable. \square

11.5.2. Monotonicity. Next, we prove the monotonicity of the integral.

Theorem 11.36. Suppose that $f, g : [a, b] \rightarrow \mathbb{R}$ are integrable and $f \leq g$. Then

$$\int_a^b f \leq \int_a^b g.$$

Proof. First suppose that $f \geq 0$ is integrable. Let P be the partition consisting of the single interval $[a, b]$. Then

$$L(f; P) = \inf_{[a,b]} f \cdot (b - a) \geq 0,$$

so

$$\int_a^b f \geq L(f; P) \geq 0.$$

If $f \geq g$, then $h = f - g \geq 0$, and the linearity of the integral implies that

$$\int_a^b f - \int_a^b g = \int_a^b h \geq 0,$$

which proves the theorem. \square

One immediate consequence of this theorem is the following simple, but useful, estimate for integrals.

Theorem 11.37. Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is integrable and

$$M = \sup_{[a,b]} f, \quad m = \inf_{[a,b]} f.$$

Then

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

Proof. Since $m \leq f \leq M$ on $[a, b]$, Theorem 11.36 implies that

$$\int_a^b m \leq \int_a^b f \leq \int_a^b M,$$

which gives the result. \square

This estimate also follows from the definition of the integral in terms of upper and lower sums, but once we've established the monotonicity of the integral, we don't need to go back to the definition.

A further consequence is the intermediate value theorem for integrals, which states that a continuous function on a compact interval is equal to its average value at some point in the interval.

Theorem 11.38. If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then there exists $x \in [a, b]$ such that

$$f(x) = \frac{1}{b-a} \int_a^b f.$$

Proof. Since f is a continuous function on a compact interval, the extreme value theorem (Theorem 7.37) implies it attains its maximum value M and its minimum value m . From Theorem 11.37,

$$m \leq \frac{1}{b-a} \int_a^b f \leq M.$$

By the intermediate value theorem (Theorem 7.44), f takes on every value between m and M , and the result follows. \square

As shown in the proof of Theorem 11.36, given linearity, monotonicity is equivalent to positivity,

$$\int_a^b f \geq 0 \quad \text{if } f \geq 0.$$

We remark that even though the upper and lower integrals aren't linear, they are monotone.

Proposition 11.39. If $f, g : [a, b] \rightarrow \mathbb{R}$ are bounded functions and $f \leq g$, then

$$U(f) \leq U(g), \quad L(f) \leq L(g).$$

Proof. From Proposition 11.1, we have for every interval $I \subset [a, b]$ that

$$\sup_I f \leq \sup_I g, \quad \inf_I f \leq \inf_I g.$$

It follows that for every partition P of $[a, b]$, we have

$$U(f; P) \leq U(g; P), \quad L(f; P) \leq L(g; P).$$

Taking the infimum of the upper inequality and the supremum of the lower inequality over P , we get that $U(f) \leq U(g)$ and $L(f) \leq L(g)$. \square

We can estimate the absolute value of an integral by taking the absolute value under the integral sign. This is analogous to the corresponding property of sums:

$$\left| \sum_{k=1}^n a_n \right| \leq \sum_{k=1}^n |a_k|.$$

Theorem 11.40. If f is integrable, then $|f|$ is integrable and

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

Proof. First, suppose that $|f|$ is integrable. Since

$$-|f| \leq f \leq |f|,$$

we get from Theorem 11.36 that

$$-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|, \quad \text{or} \quad \left| \int_a^b f \right| \leq \int_a^b |f|.$$

To complete the proof, we need to show that $|f|$ is integrable if f is integrable. For $x, y \in [a, b]$, the reverse triangle inequality gives

$$||f(x)| - |f(y)|| \leq |f(x) - f(y)|.$$

Using Proposition 11.8, we get that

$$\sup_I |f| - \inf_I |f| \leq \sup_I f - \inf_I f,$$

meaning that $\text{osc}_I |f| \leq \text{osc}_I f$. Proposition 11.25 then implies that $|f|$ is integrable if f is integrable. \square

In particular, we immediately get the following basic estimate for an integral.

Corollary 11.41. If $f : [a, b] \rightarrow \mathbb{R}$ is integrable and $M = \sup_{[a, b]} |f|$, then

$$\left| \int_a^b f \right| \leq M(b - a).$$

Finally, we prove a useful positivity result for the integral of continuous functions.

Proposition 11.42. If $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function such that $f \geq 0$ and $\int_a^b f = 0$, then $f = 0$.

Proof. Suppose for contradiction that $f(c) > 0$ for some $a \leq c \leq b$. For definiteness, assume that $a < c < b$. (The proof is similar if c is an endpoint.) Then, since f is continuous, there exists $\delta > 0$ such that

$$|f(x) - f(c)| \leq \frac{f(c)}{2} \quad \text{for } c - \delta \leq x \leq c + \delta,$$

where we choose δ small enough that $c - \delta > a$ and $c + \delta < b$. It follows that

$$f(x) = f(c) + f(x) - f(c) \geq f(c) - |f(x) - f(c)| \geq \frac{f(c)}{2}$$

for $c - \delta \leq x \leq c + \delta$. Using this inequality and the assumption that $f \geq 0$, we get

$$\int_a^b f = \int_a^{c-\delta} f + \int_{c-\delta}^{c+\delta} f + \int_{c+\delta}^b f \geq 0 + \frac{f(c)}{2} \cdot 2\delta + 0 > 0.$$

This contradiction proves the result. \square

The assumption that $f \geq 0$ is, of course, required, otherwise the integral of the function may be zero due to cancellation.

Example 11.43. The function $f : [-1, 1] \rightarrow \mathbb{R}$ defined by $f(x) = x$ is continuous and nonzero, but $\int_{-1}^1 f = 0$.

Continuity is also required; for example, the discontinuous function in Example 11.14 is nonzero, but its integral is zero.

11.5.3. Additivity. Finally, we prove additivity. This property refers to additivity with respect to the interval of integration, rather than linearity with respect to the function being integrated.

Theorem 11.44. Suppose that $f : [a, b] \rightarrow \mathbb{R}$ and $a < c < b$. Then f is Riemann integrable on $[a, b]$ if and only if it is Riemann integrable on $[a, c]$ and $[c, b]$. Moreover, in that case,

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof. Suppose that f is integrable on $[a, b]$. Then, given $\epsilon > 0$, there is a partition P of $[a, b]$ such that $U(f; P) - L(f; P) < \epsilon$. Let $P' = P \cup \{c\}$ be the refinement of P obtained by adding c to the endpoints of P . (If $c \in P$, then $P' = P$.) Then $P' = Q \cup R$ where $Q = P' \cap [a, c]$ and $R = P' \cap [c, b]$ are partitions of $[a, c]$ and $[c, b]$ respectively. Moreover,

$$U(f; P') = U(f; Q) + U(f; R), \quad L(f; P') = L(f; Q) + L(f; R).$$

It follows that

$$\begin{aligned} U(f; Q) - L(f; Q) &= U(f; P') - L(f; P') - [U(f; R) - L(f; R)] \\ &\leq U(f; P) - L(f; P) < \epsilon, \end{aligned}$$

which proves that f is integrable on $[a, c]$. Exchanging Q and R , we get the proof for $[c, b]$.

Conversely, if f is integrable on $[a, c]$ and $[c, b]$, then there are partitions Q of $[a, c]$ and R of $[c, b]$ such that

$$U(f; Q) - L(f; Q) < \frac{\epsilon}{2}, \quad U(f; R) - L(f; R) < \frac{\epsilon}{2}.$$

Let $P = Q \cup R$. Then

$$U(f; P) - L(f; P) = U(f; Q) - L(f; Q) + U(f; R) - L(f; R) < \epsilon,$$

which proves that f is integrable on $[a, b]$.

Finally, if f is integrable, then with the partitions P, Q, R as above, we have

$$\begin{aligned} \int_a^b f &\leq U(f; P) = U(f; Q) + U(f; R) \\ &< L(f; Q) + L(f; R) + \epsilon \\ &< \int_a^c f + \int_c^b f + \epsilon. \end{aligned}$$

Similarly,

$$\begin{aligned} \int_a^b f &\geq L(f; P) = L(f; Q) + L(f; R) \\ &> U(f; Q) + U(f; R) - \epsilon \\ &> \int_a^c f + \int_c^b f - \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we see that $\int_a^b f = \int_a^c f + \int_c^b f$. □

We can extend the additivity property of the integral by defining an oriented Riemann integral.

Definition 11.45. If $f : [a, b] \rightarrow \mathbb{R}$ is integrable, where $a < b$, and $a \leq c \leq b$, then

$$\int_b^a f = - \int_a^b f, \quad \int_c^c f = 0.$$

With this definition, the additivity property in Theorem 11.44 holds for all $a, b, c \in \mathbb{R}$ for which the oriented integrals exist. Moreover, if $|f| \leq M$, then the estimate in Corollary 11.41 becomes

$$\left| \int_a^b f \right| \leq M|b - a|$$

for all $a, b \in \mathbb{R}$ (even if $a \geq b$).

The oriented Riemann integral is a special case of the integral of a differential form. It assigns a value to the integral of a one-form $f dx$ on an oriented interval.

11.6. Further existence results

In this section, we prove several further useful conditions for the existences of the Riemann integral.

First, we show that changing the values of a function at finitely many points doesn't change its integrability or the value of its integral.

Proposition 11.46. Suppose that $f, g : [a, b] \rightarrow \mathbb{R}$ and $f(x) = g(x)$ except at finitely many points $x \in [a, b]$. Then f is integrable if and only if g is integrable, and in that case

$$\int_a^b f = \int_a^b g.$$

Proof. It is sufficient to prove the result for functions whose values differ at a single point, say $c \in [a, b]$. The general result then follows by repeated application of this result.

Since f, g differ at a single point, f is bounded if and only if g is bounded. If f, g are unbounded, then neither one is integrable. If f, g are bounded, we will show that f, g have the same upper and lower integrals. The reason is that their upper and lower sums differ by an arbitrarily small amount with respect to a partition that is sufficiently refined near the point where the functions differ.

Suppose that f, g are bounded with $|f|, |g| \leq M$ on $[a, b]$ for some $M > 0$. Let $\epsilon > 0$. Choose a partition P of $[a, b]$ such that

$$U(f; P) < U(f) + \frac{\epsilon}{2}.$$

Let $Q = \{I_1, \dots, I_n\}$ be a refinement of P such that $|I_k| < \delta$ for $k = 1, \dots, n$, where

$$\delta = \frac{\epsilon}{8M}.$$

Then g differs from f on at most two intervals in Q . (This could happen on two intervals if c is an endpoint of the partition.) On such an interval I_k we have

$$\left| \sup_{I_k} g - \sup_{I_k} f \right| \leq \sup_{I_k} |g| + \sup_{I_k} |f| \leq 2M,$$

and on the remaining intervals, $\sup_{I_k} g - \sup_{I_k} f = 0$. It follows that

$$|U(g; Q) - U(f; Q)| < 2M \cdot 2\delta < \frac{\epsilon}{2}.$$

Using the properties of upper integrals and refinements, we obtain that

$$U(g) \leq U(g; Q) < U(f; Q) + \frac{\epsilon}{2} \leq U(f; P) + \frac{\epsilon}{2} < U(f) + \epsilon.$$

Since this inequality holds for arbitrary $\epsilon > 0$, we get that $U(g) \leq U(f)$. Exchanging f and g , we see similarly that $U(f) \leq U(g)$, so $U(f) = U(g)$.

An analogous argument for lower sums (or an application of the result for upper sums to $-f, -g$) shows that $L(f) = L(g)$. Thus $U(f) = L(f)$ if and only if $U(g) = L(g)$, in which case $\int_a^b f = \int_a^b g$. \square

Example 11.47. The function f in Example 11.14 differs from the 0-function at one point. It is integrable and its integral is equal to 0.

The conclusion of Proposition 11.46 can fail if the functions differ at a countably infinite number of points. One reason is that we can turn a bounded function into an unbounded function by changing its values at a countably infinite number of points.

Example 11.48. Define $f : [0, 1] \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} n & \text{if } x = 1/n \text{ for } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is equal to the 0-function except on the countably infinite set $\{1/n : n \in \mathbb{N}\}$, but f is unbounded and therefore it's not Riemann integrable.

The result in Proposition 11.46 is still false, however, for bounded functions that differ at a countably infinite number of points.

Example 11.49. The Dirichlet function in Example 11.15 is bounded and differs from the 0-function on the countably infinite set of rationals, but it isn't Riemann integrable.

The Lebesgue integral is better behaved than the Riemann integral in this respect: two functions that are equal almost everywhere, meaning that they differ on a set of Lebesgue measure zero, have the same Lebesgue integrals. In particular, two functions that differ on a countable set have the same Lebesgue integrals (see Section 11.8).

The next proposition allows us to deduce the integrability of a bounded function on an interval from its integrability on slightly smaller intervals.

Proposition 11.50. Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is bounded and integrable on $[a, r]$ for every $a < r < b$. Then f is integrable on $[a, b]$ and

$$\int_a^b f = \lim_{r \rightarrow b^-} \int_a^r f.$$

Proof. Since f is bounded, $|f| \leq M$ on $[a, b]$ for some $M > 0$. Given $\epsilon > 0$, let

$$r = b - \frac{\epsilon}{4M}$$

(where we assume ϵ is sufficiently small that $r > a$). Since f is integrable on $[a, r]$, there is a partition Q of $[a, r]$ such that

$$U(f; Q) - L(f; Q) < \frac{\epsilon}{2}.$$

Then $P = Q \cup \{b\}$ is a partition of $[a, b]$ whose last interval is $[r, b]$. The boundedness of f implies that

$$\sup_{[r, b]} f - \inf_{[r, b]} f \leq 2M.$$

Therefore

$$\begin{aligned} U(f; P) - L(f; P) &= U(f; Q) - L(f; Q) + \left(\sup_{[r, b]} f - \inf_{[r, b]} f \right) \cdot (b - r) \\ &< \frac{\epsilon}{2} + 2M \cdot (b - r) = \epsilon, \end{aligned}$$

so f is integrable on $[a, b]$ by Theorem 11.23. Moreover, using the additivity of the integral, we get

$$\left| \int_a^b f - \int_a^r f \right| = \left| \int_r^b f \right| \leq M \cdot (b - r) \rightarrow 0 \quad \text{as } r \rightarrow b^-.$$

□

An obvious analogous result holds for the left endpoint.

Example 11.51. Define $f : [0, 1] \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} \sin(1/x) & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

Then f is bounded on $[0, 1]$. Furthermore, f is continuous and therefore integrable on $[r, 1]$ for every $0 < r < 1$. It follows from Proposition 11.50 that f is integrable on $[0, 1]$.

The assumption in Proposition 11.50 that f is bounded on $[a, b]$ is essential.

Example 11.52. The function $f : [0, 1] \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1/x & \text{for } 0 < x \leq 1, \\ 0 & \text{for } x = 0, \end{cases}$$

is continuous and therefore integrable on $[r, 1]$ for every $0 < r < 1$, but it's unbounded and therefore not integrable on $[0, 1]$.

As a corollary of this result and the additivity of the integral, we prove a generalization of the integrability of continuous functions to piecewise continuous functions.

Theorem 11.53. If $f : [a, b] \rightarrow \mathbb{R}$ is a bounded function with finitely many discontinuities, then f is Riemann integrable.

Proof. By splitting the interval into subintervals with the discontinuities of f at an endpoint and using Theorem 11.44, we see that it is sufficient to prove the result if f is discontinuous only at one endpoint of $[a, b]$, say at b . In that case, f is continuous and therefore integrable on any smaller interval $[a, r]$ with $a < r < b$, and Proposition 11.50 implies that f is integrable on $[a, b]$. □

Example 11.54. Define $f : [0, 2\pi] \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} \sin(1/\sin x) & \text{if } x \neq 0, \pi, 2\pi, \\ 0 & \text{if } x = 0, \pi, 2\pi. \end{cases}$$

Then f is bounded and continuous except at $x = 0, \pi, 2\pi$, so it is integrable on $[0, 2\pi]$ (see Figure 3). This function doesn't have jump discontinuities, but Theorem 11.53 still applies.

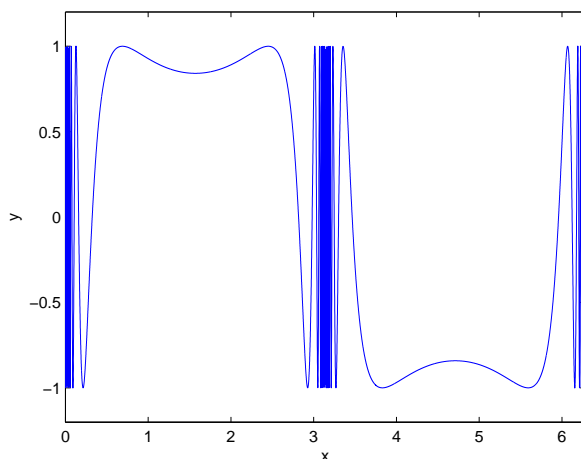


Figure 3. Graph of the Riemann integrable function $y = \sin(1/\sin x)$ in Example 11.54.

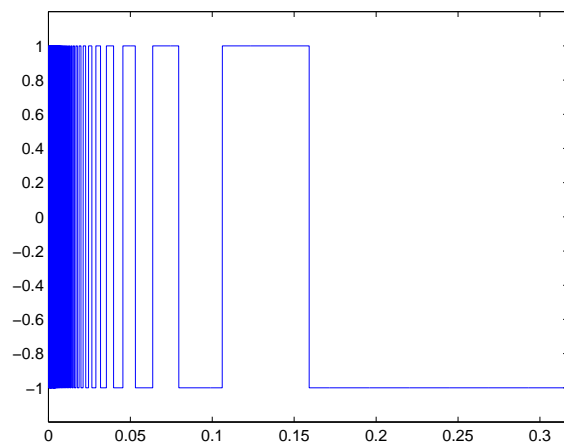


Figure 4. Graph of the Riemann integrable function $y = \operatorname{sgn}(\sin(1/x))$ in Example 11.55.

Example 11.55. Define $f : [0, 1/\pi] \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} \operatorname{sgn}[\sin(1/x)] & \text{if } x \neq 1/n\pi \text{ for } n \in \mathbb{N}, \\ 0 & \text{if } x = 0 \text{ or } x = 1/n\pi \text{ for } n \in \mathbb{N}, \end{cases}$$

where sgn is the sign function,

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Then f oscillates between 1 and -1 a countably infinite number of times as $x \rightarrow 0^+$ (see Figure 4). It has jump discontinuities at $x = 1/(n\pi)$ and an essential discontinuity at $x = 0$. Nevertheless, it is Riemann integrable. To see this, note that f is bounded on $[0, 1]$ and piecewise continuous with finitely many discontinuities on $[r, 1]$ for every $0 < r < 1$. Theorem 11.53 implies that f is Riemann integrable on $[r, 1]$, and then Theorem 11.50 implies that f is integrable on $[0, 1]$.

11.7. * Riemann sums

Instead of using upper and lower sums, we can give an equivalent definition of the Riemann integral as a limit of Riemann sums. This was, in fact, Riemann's original definition [11], which he gave in 1854 in his *Habilitationsschrift* (a kind of post-doctoral dissertation required of German academics), building on previous work of Cauchy who defined the integral for continuous functions.

It is interesting to note that the topic of Riemann's *Habilitationsschrift* was not integration theory, but Fourier series. Riemann introduced a definition of the integral along the way so that he could state his results more precisely. Many of the fundamental developments of rigorous real analysis in the nineteenth century were motivated by problems related to Fourier series and their convergence.

Upper and lower sums were introduced subsequently by Darboux, and they simplify the theory. We won't use Riemann sums here, but we will explain the equivalence of the definitions. We'll say, temporarily, that a function is Darboux integrable if it satisfies Definition 11.11.

To give Riemann's definition, we first define a tagged partition (P, C) of a compact interval $[a, b]$ to be a partition

$$P = \{I_1, I_2, \dots, I_n\}$$

of the interval together with a set

$$C = \{c_1, c_2, \dots, c_n\}$$

of points such that $c_k \in I_k$ for $k = 1, \dots, n$. (We think of the point c_k as a "tag" attached to the interval I_k .)

If $f : [a, b] \rightarrow \mathbb{R}$, then we define the Riemann sum of f with respect to the tagged partition (P, C) by

$$S(f; P, C) = \sum_{k=1}^n f(c_k)|I_k|.$$

That is, instead of using the supremum or infimum of f on the k th interval in the sum, we evaluate f at a point in the interval. Roughly speaking, a function is Riemann integrable if its Riemann sums approach the same value as the partition is refined, independently of how we choose the points $c_k \in I_k$.

As a measure of the refinement of a partition $P = \{I_1, I_2, \dots, I_n\}$, we define the mesh (or norm) of P to be the maximum length of its intervals,

$$\text{mesh}(P) = \max_{1 \leq k \leq n} |I_k| = \max_{1 \leq k \leq n} |x_k - x_{k-1}|.$$

Definition 11.56. A function $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable on $[a, b]$ if there exists a number $R \in \mathbb{R}$ with the following property: For every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|S(f; P, C) - R| < \epsilon$$

for every tagged partition (P, C) of $[a, b]$ with $\text{mesh}(P) < \delta$. In that case, $R = \int_a^b f$ is the Riemann integral of f on $[a, b]$.

Note that

$$L(f; P) \leq S(f; P, C) \leq U(f; P),$$

so the Riemann sums are “squeezed” between the upper and lower sums. The following theorem shows that the Darboux and Riemann definitions lead to the same notion of the integral, so it’s a matter of convenience which definition we adopt as our starting point.

Theorem 11.57. A function is Riemann integrable (in the sense of Definition 11.56) if and only if it is Darboux integrable (in the sense of Definition 11.11). Furthermore, in that case, the Riemann and Darboux integrals of the function are equal.

Proof. First, suppose that $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable with integral R . Then f is bounded on $[a, b]$; otherwise, it would be unbounded in some interval I_k of every partition P , and its Riemann sums with respect to P would be arbitrarily large for suitable points $c_k \in I_k$, so no $R \in \mathbb{R}$ could satisfy Definition 11.56.

Let $\epsilon > 0$. Since f is Riemann integrable, there is a partition $P = \{I_1, I_2, \dots, I_n\}$ of $[a, b]$ such that

$$|S(f; P, C) - R| < \frac{\epsilon}{2}$$

for every set of points $C = \{c_k \in I_k : k = 1, \dots, n\}$. If $M_k = \sup_{I_k} f$, then there exists $c_k \in I_k$ such that

$$M_k - \frac{\epsilon}{2(b-a)} < f(c_k).$$

It follows that

$$\sum_{k=1}^n M_k |I_k| - \frac{\epsilon}{2} < \sum_{k=1}^n f(c_k) |I_k|,$$

meaning that $U(f; P) - \epsilon/2 < S(f; P, C)$. Since $S(f; P, C) < R + \epsilon/2$, we get that

$$U(f) \leq U(f; P) < R + \epsilon.$$

Similarly, if $m_k = \inf_{I_k} f$, then there exists $c_k \in I_k$ such that

$$m_k + \frac{\epsilon}{2(b-a)} > f(c_k), \quad \sum_{k=1}^n m_k |I_k| + \frac{\epsilon}{2} > \sum_{k=1}^n f(c_k) |I_k|,$$

and $L(f; P) + \epsilon/2 > S(f; P, C)$. Since $S(f; P, C) > R - \epsilon/2$, we get that

$$L(f) \geq L(f; P) > R - \epsilon.$$

These inequalities imply that

$$L(f) + \epsilon > R > U(f) - \epsilon$$

for every $\epsilon > 0$, and therefore $L(f) \geq R \geq U(f)$. Since $L(f) \leq U(f)$, we conclude that $L(f) = R = U(f)$, so f is Darboux integrable with integral R .

Conversely, suppose that f is Darboux integrable. The main point is to show that if $\epsilon > 0$, then $U(f; P) - L(f; P) < \epsilon$ not just for some partition but for every partition whose mesh is sufficiently small.

Let $\epsilon > 0$ be given. Since f is Darboux integrable, there exists a partition Q such that

$$U(f; Q) - L(f; Q) < \frac{\epsilon}{4}.$$

Suppose that Q contains m intervals and $|f| \leq M$ on $[a, b]$. We claim that if

$$\delta = \frac{\epsilon}{8mM},$$

then $U(f; P) - L(f; P) < \epsilon$ for every partition P with $\text{mesh}(P) < \delta$.

To prove this claim, suppose that $P = \{I_1, I_2, \dots, I_n\}$ is a partition with $\text{mesh}(P) < \delta$. Let P' be the largest common refinement of P and Q , so that the endpoints of P' consist of the endpoints of P or Q . Since a, b are common endpoints of P and Q , there are at most $m - 1$ endpoints of Q that are distinct from endpoints of P . Therefore, at most $m - 1$ intervals in P contain additional endpoints of Q and are strictly refined in P' , meaning that they are the union of two or more intervals in P' .

Now consider $U(f; P) - U(f; P')$. The terms that correspond to the same, unrefined intervals in P and P' cancel. If I_k is a strictly refined interval in P , then the corresponding terms in each of the sums $U(f; P)$ and $U(f; P')$ can be estimated by $M|I_k|$ and their difference by $2M|I_k|$. There are at most $m - 1$ such intervals and $|I_k| < \delta$, so it follows that

$$U(f; P) - U(f; P') < 2(m - 1)M\delta < \frac{\epsilon}{4}.$$

Since P' is a refinement of Q , we get

$$U(f; P) < U(f; P') + \frac{\epsilon}{4} \leq U(f; Q) + \frac{\epsilon}{4} < L(f; Q) + \frac{\epsilon}{2}.$$

It follows by a similar argument that

$$L(f; P') - L(f; P) < \frac{\epsilon}{4},$$

and

$$L(f; P) > L(f; P') - \frac{\epsilon}{4} \geq L(f; Q) - \frac{\epsilon}{4} > U(f; Q) - \frac{\epsilon}{2}.$$

Since $L(f; Q) \leq U(f; Q)$, we conclude from these inequalities that

$$U(f; P) - L(f; P) < \epsilon$$

for every partition P with $\text{mesh}(P) < \delta$.

If D denotes the Darboux integral of f , then we have

$$L(f; P) \leq D \leq U(f; P), \quad L(f; P) \leq S(f; P, C) \leq U(f; P).$$

Since $U(f; P) - L(f; P) < \epsilon$ for every partition P with $\text{mesh}(P) < \delta$, it follows that

$$|S(f; P, C) - D| < \epsilon.$$

Thus, f is Riemann integrable with Riemann integral D . \square

Finally, we give a necessary and sufficient condition for Riemann integrability that was proved by Riemann himself (1854). (See [5] for further discussion.) To state the condition, we introduce some notation.

Let $f; [a, b] \rightarrow \mathbb{R}$ be a bounded function. If $P = \{I_1, I_2, \dots, I_n\}$ is a partition of $[a, b]$ and $\epsilon > 0$, let $A_\epsilon(P) \subset \{1, \dots, n\}$ be the set of indices k such that

$$\operatorname{osc}_{I_k} f = \sup_{I_k} f - \inf_{I_k} f \geq \epsilon \quad \text{for } k \in A_\epsilon(P).$$

Similarly, let $B_\epsilon(P) \subset \{1, \dots, n\}$ be the set of indices such that

$$\operatorname{osc}_{I_k} f < \epsilon \quad \text{for } k \in B_\epsilon(P).$$

That is, the oscillation of f on I_k is “large” if $k \in A_\epsilon(P)$ and “small” if $k \in B_\epsilon(P)$. We denote the sum of the lengths of the intervals in P where the oscillation of f is “large” by

$$s_\epsilon(P) = \sum_{k \in A_\epsilon(P)} |I_k|.$$

Fixing $\epsilon > 0$, we say that $s_\epsilon(P) \rightarrow 0$ as $\operatorname{mesh}(P) \rightarrow 0$ if for every $\eta > 0$ there exists $\delta > 0$ such that $\operatorname{mesh}(P) < \delta$ implies that $s_\epsilon(P) < \eta$.

Theorem 11.58. A function is Riemann integrable if and only if $s_\epsilon(P) \rightarrow 0$ as $\operatorname{mesh}(P) \rightarrow 0$ for every $\epsilon > 0$.

Proof. Let $f : [a, b] \rightarrow \mathbb{R}$ be Riemann integrable with $|f| \leq M$ on $[a, b]$.

First, suppose that the condition holds, and let $\epsilon > 0$. If P is a partition of $[a, b]$, then, using the notation above for $A_\epsilon(P)$, $B_\epsilon(P)$ and the inequality

$$0 \leq \operatorname{osc}_{I_k} f \leq 2M,$$

we get that

$$\begin{aligned} U(f; P) - L(f; P) &= \sum_{k=1}^n \operatorname{osc}_{I_k} f \cdot |I_k| \\ &= \sum_{k \in A_\epsilon(P)} \operatorname{osc}_{I_k} f \cdot |I_k| + \sum_{k \in B_\epsilon(P)} \operatorname{osc}_{I_k} f \cdot |I_k| \\ &\leq 2M \sum_{k \in A_\epsilon(P)} |I_k| + \epsilon \sum_{k \in B_\epsilon(P)} |I_k| \\ &\leq 2Ms_\epsilon(P) + \epsilon(b - a). \end{aligned}$$

By assumption, there exists $\delta > 0$ such that $s_\epsilon(P) < \epsilon$ if $\operatorname{mesh}(P) < \delta$, in which case

$$U(f; P) - L(f; P) < \epsilon(2M + b - a).$$

The Cauchy criterion in Theorem 11.23 then implies that f is integrable.

Conversely, suppose that f is integrable, and let $\epsilon > 0$ be given. If P is a partition, we can bound $s_\epsilon(P)$ from above by the difference between the upper and lower sums as follows:

$$U(f; P) - L(f; P) \geq \sum_{k \in A_\epsilon(P)} \operatorname{osc}_{I_k} f \cdot |I_k| \geq \epsilon \sum_{k \in A_\epsilon(P)} |I_k| = \epsilon s_\epsilon(P).$$

Since f is integrable, for every $\eta > 0$ there exists $\delta > 0$ such that $\text{mesh}(P) < \delta$ implies that

$$U(f; P) - L(f; P) < \epsilon \eta.$$

Therefore, $\text{mesh}(P) < \delta$ implies that

$$s_\epsilon(P) \leq \frac{1}{\epsilon} [U(f; P) - L(f; P)] < \eta,$$

which proves the result. \square

This theorem has the drawback that the necessary and sufficient condition for Riemann integrability is somewhat complicated and, in general, isn't easy to verify. In the next section, we state a simpler necessary and sufficient condition for Riemann integrability.

11.8. * The Lebesgue criterion

Although the Dirichlet function in Example 11.15 is not Riemann integrable, it is Lebesgue integrable. Its Lebesgue integral is given by

$$\int_0^1 f = 1 \cdot |A| + 0 \cdot |B|$$

where $A = [0, 1] \cap \mathbb{Q}$ is the set of rational numbers in $[0, 1]$, $B = [0, 1] \setminus \mathbb{Q}$ is the set of irrational numbers, and $|E|$ denotes the Lebesgue measure of a set E . The Lebesgue measure of a subset of \mathbb{R} is a generalization of the length of an interval which applies to more general sets. It turns out that $|A| = 0$ (as is true for any countable set of real numbers — see Example 11.60 below) and $|B| = 1$. Thus, the Lebesgue integral of the Dirichlet function is 0.

A necessary and sufficient condition for Riemann integrability can be given in terms of Lebesgue measure. To state this condition, we first define what it means for a set to have Lebesgue measure zero.

Definition 11.59. A set $E \subset \mathbb{R}$ has Lebesgue measure zero if for every $\epsilon > 0$ there is a countable collection of open intervals $\{(a_k, b_k) : k \in \mathbb{N}\}$ such that

$$E \subset \bigcup_{k=1}^{\infty} (a_k, b_k), \quad \sum_{k=1}^{\infty} (b_k - a_k) < \epsilon.$$

The open intervals in this definition are not required to be disjoint, and they may “overlap.”

Example 11.60. Every countable set $E = \{x_k \in \mathbb{R} : k \in \mathbb{N}\}$ has Lebesgue measure zero. To prove this, let $\epsilon > 0$ and for each $k \in \mathbb{N}$ define

$$a_k = x_k - \frac{\epsilon}{2^{k+2}}, \quad b_k = x_k + \frac{\epsilon}{2^{k+2}}.$$

Then $E \subset \bigcup_{k=1}^{\infty} (a_k, b_k)$ since $x_k \in (a_k, b_k)$ and

$$\sum_{k=1}^{\infty} (b_k - a_k) = \sum_{k=1}^{\infty} \frac{\epsilon}{2^{k+1}} = \frac{\epsilon}{2} < \epsilon,$$

so the Lebesgue measure of E is equal to zero. (The ‘ $\epsilon/2^k$ ’ trick used here is a common one in measure theory.)

If $E = [0, 1] \cap \mathbb{Q}$ consists of the rational numbers in $[0, 1]$, then the set $G = \bigcup_{k=1}^{\infty} (a_k, b_k)$ described above encloses the dense set of rationals in a collection of open intervals the sum of whose lengths is arbitrarily small. This set isn’t so easy to visualize. Roughly speaking, if ϵ is small and we look at a section of $[0, 1]$ at a given magnification, then we see a few of the longer intervals in G with relatively large gaps between them. Magnifying one of these gaps, we see a few more intervals with large gaps between them, magnifying those gaps, we see a few more intervals, and so on. Thus, the set G has a fractal structure, meaning that it looks similar at all scales of magnification.

In general, we have the following result, due to Lebesgue, which we state without proof.

Theorem 11.61. A function $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable if and only if it is bounded and the set of points at which it is discontinuous has Lebesgue measure zero.

For example, the set of discontinuities of the Riemann-integrable function in Example 11.14 consists of a single point $\{0\}$, which has Lebesgue measure zero. On the other hand, the set of discontinuities of the non-Riemann-integrable Dirichlet function in Example 11.15 is the entire interval $[0, 1]$, and its set of discontinuities has Lebesgue measure one.

In particular, every bounded function with a countable set of discontinuities is Riemann integrable, since such a set has Lebesgue measure zero. Riemann integrability of a function does not, however, imply that the function has only countably many discontinuities.

Example 11.62. The Cantor set C in Example 5.64 has Lebesgue measure zero. To prove this, using the same notation as in Section 5.5, we note that for every $n \in \mathbb{N}$ the set $F_n \supset C$ consists of 2^n closed intervals $I_{\mathbf{s}}$ of length $|I_{\mathbf{s}}| = 3^{-n}$. For every $\epsilon > 0$ and $\mathbf{s} \in \Sigma_n$, there is an open interval $U_{\mathbf{s}}$ of slightly larger length $|U_{\mathbf{s}}| = 3^{-n} + \epsilon 2^{-n}$ that contains $I_{\mathbf{s}}$. Then $\{U_{\mathbf{s}} : \mathbf{s} \in \Sigma_n\}$ is a cover of C by open intervals, and

$$\sum_{\mathbf{s} \in \Sigma_n} |U_{\mathbf{s}}| = \left(\frac{2}{3}\right)^n + \epsilon.$$

We can make the right-hand side as small as we wish by choosing n large enough and ϵ small enough, so C has Lebesgue measure zero.

Let $\chi_C : [0, 1] \rightarrow \mathbb{R}$ be the characteristic function of the Cantor set,

$$\chi_C(x) = \begin{cases} 1 & \text{if } x \in C, \\ 0 & \text{otherwise.} \end{cases}$$

By partitioning $[0, 1]$ into the closed intervals $\{\bar{U}_{\mathbf{s}} : \mathbf{s} \in \Sigma_n\}$ and the closures of the complementary intervals, we see similarly that the upper Riemann sums of χ_C can be made arbitrarily small, so χ_C is Riemann integrable on $[0, 1]$ with zero integral. The Riemann integrability of the function χ_C also follows from Theorem 11.61.

It is, however, discontinuous at every point of C . Thus, χ_C is an example of a Riemann integrable function with uncountably many discontinuities.

