Chapter 12

Properties and Applications of the Integral

> In the integral calculus I find much less interesting the parts that involve only substitutions, transformations, and the like, in short, the parts that involve the known skillfully applied mechanics of reducing integrals to algebraic, logarithmic, and circular functions, than I find the careful and profound study of transcendental functions that cannot be reduced to these functions. (Gauss, 1808)

12.1. The fundamental theorem of calculus

The fundamental theorem of calculus states that differentiation and integration are inverse operations in an appropriately understood sense. The theorem has two parts: in one direction, it says roughly that the integral of the derivative is the original function; in the other direction, it says that the derivative of the integral is the original function.

In more detail, the first part states that if $F : [a, b] \to \mathbb{R}$ is differentiable with integrable derivative, then

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

This result can be thought of as a continuous analog of the corresponding identity for sums of differences,

$$\sum_{k=1}^{n} (A_k - A_{k-1}) = A_n - A_0.$$

The second part states that if $f : [a, b] \to \mathbb{R}$ is continuous, then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

This is a continuous analog of the corresponding identity for differences of sums,

$$\sum_{j=1}^{k} a_j - \sum_{j=1}^{k-1} a_j = a_k.$$

The proof of the fundamental theorem consists essentially of applying the identities for sums or differences to the appropriate Riemann sums or difference quotients and proving, under appropriate hypotheses, that they converge to the corresponding integrals or derivatives.

We'll split the statement and proof of the fundamental theorem into two parts. (The numbering of the parts as I and II is arbitrary.)

12.1.1. Fundamental theorem I.

First we prove the statement about the integral of a derivative.

Theorem 12.1 (Fundamental theorem of calculus I)**.** If $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable in (a, b) with $F' = f$ where $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof. Let

$$P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\},$$

be a partition of $[a, b]$, with $x_0 = a$ and $x_n = b$. Then

$$F(b) - F(a) = \sum_{k=1}^{n} \left[F(x_k) - F(x_{k-1})\right].$$

The function F is continuous on the closed interval $[x_{k-1}, x_k]$ and differentiable in the open interval (x_{k-1}, x_k) with $F' = f$. By the mean value theorem, there exists $x_{k-1} < c_k < x_k$ such that

$$F(x_k) - F(x_{k-1}) = f(c_k)(x_k - x_{k-1}).$$

Since f is Riemann integrable, it is bounded, and

$$m_k(x_k - x_{k-1}) \le F(x_k) - F(x_{k-1}) \le M_k(x_k - x_{k-1}),$$

where

$$M_k = \sup_{[x_{k-1}, x_k]} f, \qquad m_k = \inf_{[x_{k-1}, x_k]} f.$$

Hence, $L(f; P) \le F(b) - F(a) \le U(f; P)$ for every partition P of $[a, b]$, which implies that $L(f) \le F(b) - F(a) \le U(f)$. Since f is integrable, $L(f) = U(f) = \int_a^b f$ and therefore $F(b) - F(a) = \int_a^b f$. □

In Theorem 12.1, we assume that F is continuous on the closed interval $[a, b]$ and differentiable in the open interval (a, b) where its usual two-sided derivative is defined and is equal to f. It isn't necessary to assume the existence of the right derivative of F at a or the left derivative at b, so the values of f at the endpoints are not necessarily determined by F. By Proposition 11.46, however, the integrability of f on $[a, b]$ and the value of its integral do not depend on these values, so the statement of the theorem makes sense. As a result, we'll sometimes abuse terminology and say that "F' is integrable on $[a, b]$" even if it's only defined on (a, b).

Theorem 12.1 imposes the integrability of F' as a hypothesis. Every function F that is continuously differentiable on the closed interval $[a, b]$ satisfies this condition, but the theorem remains true even if F' is a discontinuous, Riemann integrable function.

Example 12.2. Define $F : [0, 1] \to \mathbb{R}$ by

$$F(x) = \begin{cases} x^2 \sin(1/x) & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

Then F is continuous on $[0, 1]$ and, by the product and chain rules, differentiable in $(0, 1]$. It is also differentiable — but not continuously differentiable — at 0, with $F'(0^+) = 0$. Thus,

$$F'(x) = \begin{cases} -\cos(1/x) + 2x \sin(1/x) & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

The derivative F' is bounded on $[0, 1]$ and discontinuous only at one point ($x = 0$), so Theorem 11.53 implies that F' is integrable on $[0, 1]$. This verifies all of the hypotheses in Theorem 12.1, and we conclude that

$$\int_0^1 F'(x)\,dx = \sin 1.$$

There are, however, differentiable functions whose derivatives are unbounded or so discontinuous that they aren't Riemann integrable.

Example 12.3. Define $F : [0, 1] \to \mathbb{R}$ by $F(x) = \sqrt{x}$. Then F is continuous on $[0, 1]$ and differentiable in $(0, 1]$, with

$$F'(x) = \frac{1}{2\sqrt{x}} \qquad \text{for } 0 < x \leq 1.$$

This function is unbounded, so F' is not Riemann integrable on $[0, 1]$, however we define its value at 0, and Theorem 12.1 does not apply.

We can interpret the integral of F' on $[0, 1]$ as an improper Riemann integral (as is discussed further in Section 12.4). The function F is continuously differentiable on $[\epsilon, 1]$ for every $0 < \epsilon < 1$, so

$$\int_\epsilon^1 \frac{1}{2\sqrt{x}}\,dx = 1 - \sqrt{\epsilon}.$$

Thus, we get the improper integral

$$\lim_{\epsilon \to 0^+} \int_\epsilon^1 \frac{1}{2\sqrt{x}}\,dx = 1.$$

The construction of a function with a bounded, non-integrable derivative is more involved. It's not sufficient to give a function with a bounded derivative that is discontinuous at finitely many points, as in Example 12.2, because such a function is Riemann integrable. Rather, one has to construct a differentiable function whose derivative is discontinuous on a set of nonzero Lebesgue measure. Abbott [**1**] gives an example.

Finally, we remark that Theorem 12.1 remains valid for the oriented Riemann integral, since exchanging a and b reverses the sign of both sides.

12.1.2. Fundamental theorem of calculus II. Next, we prove the other direction of the fundamental theorem. We will use the following result, of independent interest, which states that the average of a continuous function on an interval approaches the value of the function as the length of the interval shrinks to zero. The proof uses a common trick of taking a constant inside an average.

Theorem 12.4. *Suppose that $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ and continuous at a. Then*

$$\lim_{h\to 0^+} \frac{1}{h} \int_a^{a+h} f(x)\,dx = f(a).$$

Proof. If k is a constant, then we have

$$k = \frac{1}{h} \int_a^{a+h} k\,dx.$$

(That is, the average of a constant is equal to the constant.) We can therefore write

$$\frac{1}{h} \int_a^{a+h} f(x)\,dx - f(a) = \frac{1}{h} \int_a^{a+h} [f(x) - f(a)]\,dx.$$

Let $\epsilon > 0$. Since f is continuous at a, there exists $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \quad \text{for} \quad a \le x < a + \delta.$$

It follows that if $0 < h < \delta$, then

$$\left| \frac{1}{h} \int_a^{a+h} f(x)\,dx - f(a) \right| \le \frac{1}{h} \cdot \sup_{a \le a \le a+h} |f(x) - f(a)| \cdot h \le \epsilon,$$

which proves the result. □

A similar proof shows that if f is continuous at b, then

$$\lim_{h\to 0^+} \frac{1}{h} \int_{b-h}^{b} f = f(b),$$

and if f is continuous at $a < c < b$, then

$$\lim_{h\to 0^+} \frac{1}{2h} \int_{c-h}^{c+h} f = f(c).$$

More generally, if f is continuous at c and $\{I_h : h > 0\}$ is any collection of intervals with $c \in I_h$ and $|I_h| \to 0$ as $h \to 0^+$, then

$$\lim_{h\to 0^+} \frac{1}{|I_h|} \int_{I_h} f = f(c).$$

The assumption in Theorem 12.4 that f is continuous at the point about which we take the averages is essential.

Example 12.5. Let $f : \mathbb{R} \to \mathbb{R}$ be the sign function

$$f(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Then

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h f(x)\,dx = 1, \qquad \lim_{h\to 0^+} \frac{1}{h}\int_{-h}^0 f(x)\,dx = -1,$$

and neither limit is equal to $f(0)$. In this example, the limit of the symmetric averages

$$\lim_{h\to 0^+} \frac{1}{2h}\int_{-h}^h f(x)\,dx = 0$$

is equal to $f(0)$, but this equality doesn't hold if we change $f(0)$ to a nonzero value (for example, if $f(x) = 1$ for $x \geq 0$ and $f(x) = -1$ for $x < 0$) since the limit of the symmetric averages is still 0.

The second part of the fundamental theorem follows from this result and the fact that the difference quotients of F are averages of f.

Theorem 12.6 (Fundamental theorem of calculus II)**.** Suppose that $f : [a, b] \to \mathbb{R}$ is integrable and $F : [a, b] \to \mathbb{R}$ is defined by

$$F(x) = \int_a^x f(t)\,dt.$$

Then F is continuous on $[a, b]$. Moreover, if f is continuous at $a \leq c \leq b$, then F is differentiable at c and $F'(c) = f(c)$.

Proof. First, note that Theorem 11.44 implies that f is integrable on $[a, x]$ for every $a \leq x \leq b$, so F is well-defined. Since f is Riemann integrable, it is bounded, and $|f| \leq M$ for some $M \geq 0$. It follows that

$$|F(x+h) - F(x)| = \left|\int_x^{x+h} f(t)\,dt\right| \leq M|h|,$$

which shows that F is continuous on $[a, b]$ (in fact, Lipschitz continuous).

Moreover, we have

$$\frac{F(c+h) - F(c)}{h} = \frac{1}{h}\int_c^{c+h} f(t)\,dt.$$

It follows from Theorem 12.4 that if f is continuous at c, then F is differentiable at c with

$$F'(c) = \lim_{h\to 0}\left[\frac{F(c+h) - F(c)}{h}\right] = \lim_{h\to 0} \frac{1}{h}\int_c^{c+h} f(t)\,dt = f(c),$$

where we use the appropriate right or left limit at an endpoint. □

The assumption that f is continuous is needed to ensure that F is differentiable.

Example 12.7. If

$$f(x) = \begin{cases} 1 & \text{for } x \geq 0, \\ 0 & \text{for } x < 0, \end{cases}$$

then

$$F(x) = \int_0^x f(t)\, dt = \begin{cases} x & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

The function F is continuous but not differentiable at $x = 0$, where f is discontinuous, since the left and right derivatives of F at 0, given by $F'(0^-) = 0$ and $F'(0^+) = 1$, are different.

12.2. Consequences of the fundamental theorem

The first part of the fundamental theorem, Theorem 12.1, is the basic tool for the exact evaluation of integrals. It allows us to compute the integral of of a function f if we can find an antiderivative; that is, a function F such that $F' = f$. There is no systematic procedure for finding antiderivatives. Moreover, an antiderivative of an elementary function (constructed from power, trigonometric, and exponential functions and their inverses) need not be — and often isn't — expressible in terms of elementary functions. By contrast, the rules of differentiation provide a mechanical algorithm for the computation of the derivative of any function formed from elementary functions by algebraic operations and compositions.

Example 12.8. For $p = 0, 1, 2, \dots$, we have

$$\frac{d}{dx}\left[\frac{1}{p+1} x^{p+1}\right] = x^p,$$

and it follows that

$$\int_0^1 x^p\, dx = \frac{1}{p+1}.$$

Example 12.9. We can use the fundamental theorem to evaluate certain limits of sums. For example,

$$\lim_{n\to\infty}\left[\frac{1}{n^{p+1}} \sum_{k=1}^n k^p\right] = \frac{1}{p+1},$$

since the sum on the left-hand side is the upper sum of x^p on a partition of $[0, 1]$ into n intervals of equal length. Example 11.27 illustrates this result explicitly for $p = 2$.

Two important general consequences of the first part of the fundamental theorem are integration by parts and substitution (or change of variable), which come from inverting the product rule and chain rule for derivatives, respectively.

Theorem 12.10 (Integration by parts)**.** *Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable in (a, b), and f', g' are integrable on $[a, b]$. Then*

$$\int_a^b f g'\, dx = f(b)g(b) - f(a)g(a) - \int_a^b f' g\, dx.$$

Proof. The function fg is continuous on $[a,b]$ and, by the product rule, differentiable in (a,b) with derivative

$$(fg)' = fg' + f'g.$$

Since f, g, f', g' are integrable on $[a,b]$, Theorem 11.35 implies that fg', $f'g$, and $(fg)'$, are integrable. From Theorem 12.1, we get that

$$\int_a^b fg'\,dx + \int_a^b f'g\,dx = \int_a^b (fg)'\,dx = f(b)g(b) - f(a)g(a),$$

which proves the result. □

Integration by parts says that we can move a derivative from one factor in an integral onto the other factor, with a change of sign and the appearance of a boundary term. The product rule for derivatives expresses the derivative of a product in terms of the derivatives of the factors. By contrast, integration by parts doesn't give an explicit expression for the integral of a product, it simply replaces one integral by another. This can sometimes be used transform an integral into an integral that is easier to evaluate, but the importance of integration by parts goes far beyond its use as an integration technique.

Example 12.11. For $n = 0, 1, 2, 3, \dots$, let

$$I_n(x) = \int_0^x t^n e^{-t}\,dt.$$

If $n \geq 1$, then integration by parts with $f(t) = t^n$ and $g'(t) = e^{-t}$ gives

$$I_n(x) = -x^n e^{-x} + n\int_0^x t^{n-1}e^{-t}\,dt = -x^n e^{-x} + nI_{n-1}(x).$$

Also, by the fundamental theorem of calculus,

$$I_0(x) = \int_0^x e^{-t}\,dt = 1 - e^{-x}.$$

It then follows by induction that

$$I_n(x) = n!\left[1 - e^{-x}\sum_{k=0}^{n}\frac{x^k}{k!}\right].$$

Since $x^k e^{-x} \to 0$ as $x \to \infty$ for every $k = 0, 1, 2, \dots$, we get the improper integral

$$\int_0^\infty t^n e^{-t}\,dt = \lim_{r\to\infty}\int_0^r t^n e^{-t}\,dt = n!$$

This formula suggests an extension of the factorial function to complex numbers $z \in \mathbb{C}$, called the Gamma function, which is defined for $\Re z > 0$ by the improper, complex-valued integral

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt.$$

In particular, $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. The Gamma function is an important special function, which is studied further in complex analysis.

Next we consider the change of variable formula for integrals.

Theorem 12.12 (Change of variable)**.** Suppose that $g : I \to \mathbb{R}$ differentiable on an open interval I and g' is integrable on I. Let $J = g(I)$. If $f : J \to \mathbb{R}$ continuous, then for every $a, b \in I$,

$$\int_a^b f\left(g(x)\right) g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Proof. For $x \in J$, let

$$F(x) = \int_{g(a)}^{x} f(u)\, du.$$

Since f is continuous, Theorem 12.6 implies that F is differentiable in J with $F' = f$. The chain rule implies that the composition $F \circ g : I \to \mathbb{R}$ is differentiable in I, with

$$(F \circ g)'(x) = f\left(g(x)\right) g'(x).$$

This derivative is integrable on $[a, b]$ since $f \circ g$ is continuous and g' is integrable. Theorem 12.1, the definition of F, and the additivity of the integral then imply that

$$\begin{aligned}\int_a^b f\left(g(x)\right) g'(x)\, dx &= \int_a^b (F \circ g)'\, dx \\ &= F\left(g(b)\right) - F\left(g(a)\right) \\ &= \int_{g(a)}^{g(b)} F'(u)\, du,\end{aligned}$$

which proves the result. □

There is no assumption in this theorem that g is invertible, but we often use the theorem in that case. A continuous function maps an interval to an interval, and it is one-to-one if and only if it is strictly monotone. An increasing function preserves the orientation of the interval, while a decreasing function reverses it, in which case the integrals in the previous theorem are understood as oriented integrals.

This result can also be formulated in terms of non-oriented integrals. Suppose that $g : I \to J$ is one-to-one and onto from an interval $I = [a, b]$ to an interval $J = g(I) = [c, d]$ where $c = g(a)$, $d = g(b)$ if g is increasing, and $c = g(b)$, $d = g(a)$ if g is decreasing, then

$$\int_{g(I)} f(u)\, du = \int_I (f \circ g)(x) |g'(x)|\, dx.$$

In this identity, both integrals are over positively oriented intervals and we include an absolute value in the Jacobian factor $|g'(x)|$. If $g' \geq 0$, then this identity is the same as the oriented form, while if $g' \leq 0$, then

$$\begin{aligned}
\int_I (f \circ g)(x)|g'(x)|\,dx &= \int_a^b (f \circ g)(x)[-g'(x)]\,dx \\
&= -\int_{g(a)}^{g(b)} f(u)\,du \\
&= \int_{g(b)}^{g(a)} f(u)\,du \\
&= \int_{g(I)} f(u)\,du.
\end{aligned}$$

Example 12.13. For every $a > 0$, the increasing, differentiable function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^3$ maps $(-a, a)$ one-to-one and onto $(-a^3, a^3)$ and preserves orientation. Thus, if $f : [-a, a] \to \mathbb{R}$ is continuous,

$$\int_{-a}^{a} f(x^3) \cdot 3x^2\,dx = \int_{-a^3}^{a^3} f(u)\,du.$$

The decreasing, differentiable function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = -x^3$ maps $(-a, a)$ one-to-one and onto $(-a^3, a^3)$ and reverses orientation. Thus,

$$\int_{-a}^{a} f(-x^3) \cdot (-3x^2)\,dx = \int_{a^3}^{-a^3} f(u)\,du = -\int_{-a^3}^{a^3} f(u)\,du.$$

The non-monotone, differentiable function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$ maps $(-a, a)$ onto $[0, a^2)$. It is two-to-one, except at $x = 0$. The change of variables formula gives

$$\int_{-a}^{a} f(x^2) \cdot 2x\,dx = \int_{a^2}^{a^2} f(u)\,du = 0.$$

The contributions to the original integral from $[0, a]$ and $[-a, 0]$ cancel since the integrand is an odd function of x.

One consequence of the second part of the fundamental theorem, Theorem 12.6, is that every continuous function has an antiderivative, even if it can't be expressed explicitly in terms of elementary functions. This provides a way to define transcendental functions as integrals of elementary functions.

Example 12.14. One way to define the natural logarithm $\log : (0, \infty) \to \mathbb{R}$ in terms of algebraic functions is as the integral

$$\log x = \int_1^x \frac{1}{t}\,dt.$$

This integral is well-defined for every $0 < x < \infty$ since $1/t$ is continuous on the interval $[1, x]$ if $x > 1$, or $[x, 1]$ if $0 < x < 1$. The usual properties of the logarithm follow from this representation. We have $(\log x)' = 1/x$ by definition; and, for example, by making the substitution $s = xt$ in the second integral in the following equation, when $dt/t = ds/s$, we get

$$\log x + \log y = \int_1^x \frac{1}{t}\,dt + \int_1^y \frac{1}{t}\,dt = \int_1^x \frac{1}{t}\,dt + \int_x^{xy} \frac{1}{s}\,ds = \int_1^{xy} \frac{1}{t}\,dt = \log(xy).$$

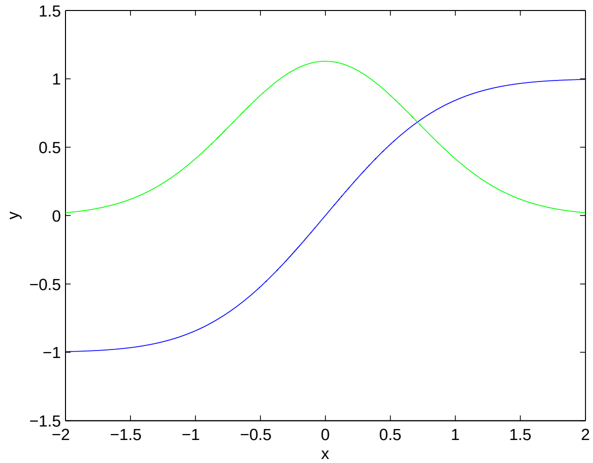

Figure 1. Graphs of the error function $y = F(x)$ (blue) and its derivative, the Gaussian function $y = f(x)$ (green), from Example 12.15.

We can also define many non-elementary functions as integrals.

Example 12.15. The error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

is an anti-derivative on $\mathbb{R}$ of the Gaussian function

$$f(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}.$$

The error function isn't expressible in terms of elementary functions. Nevertheless, it is defined as a limit of Riemann sums for the integral. Figure 1 shows the graphs of f and F. The name "error function" comes from the fact that the probability of a Gaussian random variable deviating by more than a given amount from its mean can be expressed in terms of F. Error functions also arise in other applications; for example, in modeling diffusion processes such as heat flow.

Example 12.16. The Fresnel sine function S is defined by

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt.$$

The function S is an antiderivative of $\sin(\pi t^2/2)$ on $\mathbb{R}$ (see Figure 2), but it can't be expressed in terms of elementary functions. Fresnel integrals arise, among other places, in analysing the diffraction of waves, such as light waves. From the perspective of complex analysis, they are closely related to the error function through the Euler formula $e^{i\theta} = \cos\theta + i \sin\theta$.

Discontinuous functions may or may not have an antiderivative and typically don't. Darboux proved that every function $f : (a, b) \to \mathbb{R}$ that is the derivative of a function $F : (a, b) \to \mathbb{R}$, where $F' = f$ at all points of (a, b), has the intermediate

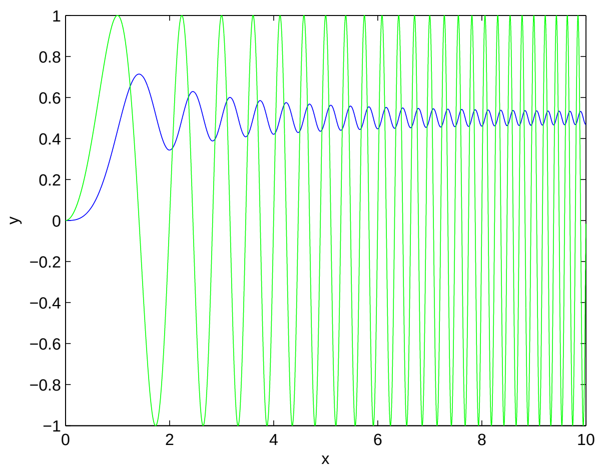

Figure 2. Graphs of the Fresnel integral $y = S(x)$ (blue) and its derivative $y = \sin(\pi x^2/2)$ (green) from Example 12.16.

value property. That is, for all c, d such that if $a < c < d < b$ and all y between $f(c)$ and $f(d)$, there exists an x between c and d such that $f(x) = y$. A continuous derivative has this property by the intermediate value theorem, but a discontinuous derivative also has it. Thus, discontinuous functions without the intermediate value property, such as ones with a jump discontinuity, don't have an antiderivative. For example, the function F in Example 12.7 is not an antiderivative of the step function f on $\mathbb{R}$ since it isn't differentiable at 0.

In dealing with functions that are not continuously differentiable, it turns out to be more useful to abandon the idea of a derivative that is defined pointwise everywhere (pointwise values of discontinuous functions are somewhat arbitrary) and introduce the notion of a weak derivative. We won't define or study weak derivatives here.

12.3. Integrals and sequences of functions

A fundamental question that arises throughout analysis is the validity of an exchange in the order of limits. Some sort of condition is always required.

In this section, we consider the question of when the convergence of a sequence of functions $f_n \to f$ implies the convergence of their integrals $\int f_n \to \int f$. Here, we exchange a limit of a sequence of functions with a limit of the Riemann sums that define their integrals. The two types of convergence we'll discuss are pointwise and uniform convergence, which are defined in Chapter 9.

As we show first, the Riemann integral is well-behaved with respect to uniform convergence. The drawback to uniform convergence is that it's a strong form of convergence, and we often want to use a weaker form, such as pointwise convergence, in which case the Riemann integral may not be suitable.

12.3.1. Uniform convergence. The uniform limit of continuous functions is continuous and therefore integrable. The next result shows, more generally, that the uniform limit of integrable functions is integrable. Furthermore, the limit of the integrals is the integral of the limit.

Theorem 12.17. Suppose that $f_n : [a, b] \to \mathbb{R}$ is Riemann integrable for each $n \in \mathbb{N}$ and $f_n \to f$ uniformly on $[a, b]$ as $n \to \infty$. Then $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

Proof. The main statement we need to prove is that f is integrable. Let $\epsilon > 0$. Since $f_n \to f$ uniformly, there is an $N \in \mathbb{N}$ such that if $n > N$ then

$$f_n(x) - \frac{\epsilon}{b-a} < f(x) < f_n(x) + \frac{\epsilon}{b-a} \qquad \text{for all } a \le x \le b.$$

It follows from Proposition 11.39 that

$$L\left(f_n - \frac{\epsilon}{b-a}\right) \le L(f), \qquad U(f) \le U\left(f_n + \frac{\epsilon}{b-a}\right).$$

Since f_n is integrable and upper integrals are greater than lower integrals, we get that

$$\int_a^b f_n - \epsilon \le L(f) \le U(f) \le \int_a^b f_n + \epsilon$$

for all $n > N$, which implies that

$$0 \le U(f) - L(f) \le 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we conclude that $L(f) = U(f)$, so f is integrable. Moreover, it follows that for all $n > N$ we have

$$\left| \int_a^b f_n - \int_a^b f \right| \le \epsilon,$$

which shows that $\int_a^b f_n \to \int_a^b f$ as $n \to \infty$. □

Alternatively, once we know that the uniform limit of integrable functions is integrable, the convergence of the integrals follows directly from the estimate

$$\left| \int_a^b f_n - \int_a^b f \right| = \left| \int_a^b (f_n - f) \right| \le \sup_{x\in[a,b]} |f_n(x) - f(x)| \cdot (b-a) \to 0 \qquad \text{as } n \to \infty.$$

Example 12.18. The function $f_n : [0, 1] \to \mathbb{R}$ defined by

$$f_n(x) = \frac{n + \cos x}{ne^x + \sin x}$$

converges uniformly on $[0, 1]$ to $f(x) = e^{-x}$ since, for $0 \le x \le 1$,

$$\left| \frac{n + \cos x}{ne^x + \sin x} - e^{-x} \right| = \left| \frac{\cos x - e^{-x} \sin x}{ne^x + \sin x} \right| \le \frac{1}{n}.$$

It follows that

$$\lim_{n\to\infty} \int_0^1 \frac{n + \cos x}{ne^x + \sin x}\, dx = \int_0^1 e^{-x}\, dx = 1 - \frac{1}{e}.$$

Example 12.19. Every power series

$$f(x) = a_0 + a_1 x + a^2 x^2 + \dots + a_n x^n + \dots$$

with radius of convergence $R > 0$ converges uniformly on compact intervals inside the interval $|x| < R$, so we can integrate it term-by-term to get

$$\int_0^x f(t)\,dt = a_0 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_2 x^3 + \dots + \frac{1}{n+1} a_n x^{n+1} + \dots \qquad \text{for } |x| < R.$$

Example 12.20. If we integrate the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots \qquad \text{for } |x| < 1,$$

we get a power series for log,

$$\log\left(\frac{1}{1-x}\right) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 \dots + \frac{1}{n}x^n + \dots \qquad \text{for } |x| < 1.$$

For instance, taking $x = 1/2$, we get the rapidly convergent series

$$\log 2 = \sum_{n=1}^{\infty} \frac{1}{n 2^n}$$

for the irrational number $\log 2 \approx 0.6931$. This series was known and used by Euler. For comparison, the alternating harmonic series in Example 12.46 also converges to $\log 2$, but it does so extremely slowly and would be a poor choice for computing a numerical approximation.

Although we can integrate uniformly convergent sequences, we cannot in general differentiate them. In fact, it's often easier to prove results about the convergence of derivatives by using results about the convergence of integrals, together with the fundamental theorem of calculus. The following theorem provides sufficient conditions for $f_n \to f$ to imply that $f_n' \to f'$.

Theorem 12.21. *Let $f_n : (a, b) \to \mathbb{R}$ be a sequence of differentiable functions whose derivatives $f_n' : (a, b) \to \mathbb{R}$ are integrable on (a, b). Suppose that $f_n \to f$ pointwise and $f_n' \to g$ uniformly on (a, b) as $n \to \infty$, where $g : (a, b) \to \mathbb{R}$ is continuous. Then $f : (a, b) \to \mathbb{R}$ is continuously differentiable on (a, b) and $f' = g$.*

Proof. Choose some point $a < c < b$. Since f_n' is integrable, the fundamental theorem of calculus, Theorem 12.1, implies that

$$f_n(x) = f_n(c) + \int_c^x f_n' \qquad \text{for } a < x < b.$$

Since $f_n \to f$ pointwise and $f_n' \to g$ uniformly on $[a, x]$, we find that

$$f(x) = f(c) + \int_c^x g.$$

Since g is continuous, the other direction of the fundamental theorem, Theorem 12.6, implies that f is differentiable in (a, b) and $f' = g$. □

In particular, this theorem shows that the limit of a uniformly convergent sequence of continuously differentiable functions whose derivatives converge uniformly is also continuously differentiable.

The key assumption in Theorem 12.21 is that the derivatives f_n' converge uniformly, not just pointwise; the result is false if we only assume pointwise convergence of the f_n'. In the proof of the theorem, we only use the assumption that $f_n(x)$ converges at a single point $x = c$. This assumption together with the assumption that $f_n' \to g$ uniformly implies that $f_n \to f$ uniformly, where

$$f(x) = \lim_{n\to\infty} f_n(c) + \int_c^x g.$$

Thus, the theorem remains true if we replace the assumption that $f_n \to f$ pointwise on (a, b) by the weaker assumption that $\lim_{n\to\infty} f_n(c)$ exists for some $c \in (a, b)$. This isn't an important change, however, because the restrictive assumption in the theorem is the uniform convergence of the derivatives f_n', not the pointwise (or uniform) convergence of the functions f_n.

The assumption that $g = \lim f_n'$ is continuous is needed to show the differentiability of f by the fundamental theorem, but the result remains true even if g isn't continuous. In that case, however, a different — and more complicated — proof is required, which is given in Theorem 9.18.

12.3.2. Pointwise convergence. On its own, the pointwise convergence of functions is never sufficient to imply convergence of their integrals.

Example 12.22. For $n \in \mathbb{N}$, define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} n & \text{if } 0 < x < 1/n, \\ 0 & \text{if } x = 0 \text{ or } 1/n \le x \le 1. \end{cases}$$

Then $f_n \to 0$ pointwise on $[0, 1]$ but

$$\int_0^1 f_n = 1$$

for every $n \in \mathbb{N}$. By slightly modifying these functions to

$$f_n(x) = \begin{cases} n^2 & \text{if } 0 < x < 1/n, \\ 0 & \text{if } x = 0 \text{ or } 1/n \le x \le 1, \end{cases}$$

we get a sequence that converges pointwise to 0 but whose integrals diverge to ∞. The fact that the f_n are discontinuous is not important; we could replace the step functions by continuous "tent" functions or smooth "bump" functions.

The behavior of the integral under pointwise convergence in the previous example is unavoidable whatever definition of the integral one uses. A more serious defect of the Riemann integral is that the pointwise limit of Riemann integrable functions needn't be Riemann integrable at all, even if it is bounded.

Example 12.23. Let $\{r_k : k \in \mathbb{N}\}$ be an enumeration of the rational numbers in $[0, 1]$ and define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x = r_k \text{ for some } 1 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Then each f_n is Riemann integrable since it differs from the zero function at finitely many points. However, $f_n \to f$ pointwise on $[0, 1]$ to the Dirichlet function f, which is not Riemann integrable.

This is another place where the Lebesgue integral has better properties than the Riemann integral. The pointwise (or pointwise almost everywhere) limit of Lebesgue measurable functions is Lebesgue measurable. As Example 12.22 shows, we still need conditions to ensure the convergence of the integrals, but there are quite simple and general conditions for the Lebesgue integral (such as the monotone convergence and dominated convergence theorems).

12.4. Improper Riemann integrals

The Riemann integral is only defined for a bounded function on a compact interval (or a finite union of such intervals). Nevertheless, we frequently want to integrate unbounded functions or functions on an infinite interval. One way to interpret such integrals is as a limit of Riemann integrals; these limits are called improper Riemann integrals.

12.4.1. Improper integrals.

First, we define the improper integral of a function that fails to be integrable at one endpoint of a bounded interval.

Definition 12.24. Suppose that $f : (a, b] \to \mathbb{R}$ is integrable on $[c, b]$ for every $a < c < b$. Then the improper integral of f on $[a, b]$ is

$$\int_a^b f = \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^b f.$$

The improper integral converges if this limit exists (as a finite real number), otherwise it diverges. Similarly, if $f : [a, b) \to \mathbb{R}$ is integrable on $[a, c]$ for every $a < c < b$, then

$$\int_a^b f = \lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f.$$

We use the same notation to denote proper and improper integrals; it should be clear from the context which integrals are proper Riemann integrals (i.e., ones given by Definition 11.11) and which are improper. If f is Riemann integrable on $[a, b]$, then Proposition 11.50 shows that its improper and proper integrals agree, but an improper integral may exist even if f isn't integrable.

Example 12.25. If $p > 0$, then the integral

$$\int_0^1 \frac{1}{x^p}\, dx$$

isn't defined as a Riemann integral since $1/x^p$ is unbounded on $(0, 1]$. The corresponding improper integral is

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{\epsilon\to 0^+}\int_\epsilon^1 \frac{1}{x^p}\,dx.$$

For $p \neq 1$, we have

$$\int_\epsilon^1 \frac{1}{x^p}\,dx = \frac{1-\epsilon^{1-p}}{1-p},$$

so the improper integral converges if $0 < p < 1$, with

$$\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{p-1},$$

and diverges to ∞ if $p > 1$. The integral also diverges (more slowly) to ∞ if $p = 1$ since

$$\int_\epsilon^1 \frac{1}{x}\,dx = \log\frac{1}{\epsilon}.$$

Thus, we get a convergent improper integral if the integrand $1/x^p$ does not grow too rapidly as $x \to 0^+$ (slower than $1/x$).

We define improper integrals on an unbounded interval as limits of integrals on bounded intervals.

Definition 12.26. Suppose that $f : [a,\infty) \to \mathbb{R}$ is integrable on $[a, r]$ for every $r > a$. Then the improper integral of f on $[a,\infty)$ is

$$\int_a^\infty f = \lim_{r\to\infty}\int_a^r f.$$

Similarly, if $f : (-\infty, b] \to \mathbb{R}$ is integrable on $[r, b]$ for every $r < b$, then

$$\int_{-\infty}^b f = \lim_{r\to\infty}\int_{-r}^b f.$$

Let's consider the convergence of the integral of the power function in Example 12.25 at infinity rather than at zero.

Example 12.27. Suppose that $p > 0$. The improper integral

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{r\to\infty}\int_1^r \frac{1}{x^p}\,dx = \lim_{r\to\infty}\left(\frac{r^{1-p}-1}{1-p}\right)$$

converges to $1/(p-1)$ if $p > 1$ and diverges to ∞ if $0 < p < 1$. It also diverges (more slowly) if $p = 1$ since

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{r\to\infty}\int_1^r \frac{1}{x}\,dx = \lim_{r\to\infty}\log r = \infty.$$

Thus, we get a convergent improper integral if the integrand $1/x^p$ decays sufficiently rapidly as $x \to \infty$ (faster than $1/x$).

A divergent improper integral may diverge to ∞ (or $-\infty$) as in the previous examples, or — if the integrand changes sign — it may oscillate.

Example 12.28. Define $f : [0, \infty) \to \mathbb{R}$ by

$$f(x) = (-1)^n \qquad \text{for } n \le x < n+1 \text{ where } n = 0, 1, 2, \dots.$$

Then $0 \le \int_0^r f \le 1$ and

$$\int_0^n f = \begin{cases} 1 & \text{if } n \text{ is an odd integer,} \\ 0 & \text{if } n \text{ is an even integer.} \end{cases}$$

Thus, the improper integral $\int_0^\infty f$ doesn't converge.

More general improper integrals may be defined as finite sums of improper integrals of the previous forms. For example, if $f : [a, b] \setminus \{c\} \to \mathbb{R}$ is integrable on closed intervals not including $a < c < b$, then

$$\int_a^b f = \lim_{\delta \to 0^+} \int_a^{c-\delta} f + \lim_{\epsilon \to 0^+} \int_{c+\epsilon}^b f;$$

and if $f : \mathbb{R} \to \mathbb{R}$ is integrable on every compact interval, then

$$\int_{-\infty}^\infty f = \lim_{s \to \infty} \int_{-s}^c f + \lim_{r \to \infty} \int_c^r f,$$

where we split the integral at an arbitrary point $c \in \mathbb{R}$. Note that each limit is required to exist separately.

Example 12.29. The improper Riemann integral

$$\int_0^\infty \frac{1}{x^p}\, dx = \lim_{\epsilon \to 0^+} \int_\epsilon^1 \frac{1}{x^p}\, dx + \lim_{r \to \infty} \int_1^r \frac{1}{x^p}\, dx$$

does not converge for any $p \in \mathbb{R}$, since the integral either diverges at 0 (if $p \ge 1$) or at infinity (if $p \le 1$).

Example 12.30. If $f : [0, 1] \to \mathbb{R}$ is continuous and $0 < c < 1$, then we define as an improper integral

$$\int_0^1 \frac{f(x)}{|x-c|^{1/2}}\, dx = \lim_{\delta \to 0^+} \int_0^{c-\delta} \frac{f(x)}{|x-c|^{1/2}}\, dx + \lim_{\epsilon \to 0^+} \int_{c+\epsilon}^1 \frac{f(x)}{|x-c|^{1/2}}\, dx.$$

Example 12.31. Consider the following integral, called a Frullani integral,

$$I = \int_0^\infty \frac{f(ax) - f(bx)}{x}\, dx,$$

where $0 < a < b$ and $f : [0, \infty) \to \mathbb{R}$ is a continuous function whose limit as $x \to \infty$ exists. We write this limit as

$$f(\infty) = \lim_{x \to \infty} f(x).$$

We interpret the integral as an improper integral $I = I_1 + I_2$ where

$$I_1 = \lim_{\epsilon \to 0^+} \int_\epsilon^1 \frac{f(ax) - f(bx)}{x}\, dx, \qquad I_2 = \lim_{r \to \infty} \int_1^r \frac{f(ax) - f(bx)}{x}\, dx.$$

Consider I_1. After making the substitutions $s = ax$ and $t = bx$ and using the additivity property of the integral, we get that

$$I_1 = \lim_{\epsilon \to 0^+} \left(\int_{\epsilon a}^a \frac{f(s)}{s}\, ds - \int_{\epsilon b}^b \frac{f(t)}{t}\, dt \right) = \lim_{\epsilon \to 0^+} \left(\int_{\epsilon a}^{\epsilon b} \frac{f(t)}{t}\, dt \right) - \int_a^b \frac{f(t)}{t}\, dt.$$

To evaluate the limit, we write

$$\begin{aligned}\int_{\epsilon a}^{\epsilon b} \frac{f(t)}{t}\,dt &= \int_{\epsilon a}^{\epsilon b} \frac{f(t)-f(0)}{t}\,dt + f(0)\int_{\epsilon a}^{\epsilon b} \frac{1}{t}\,dt \\ &= \int_{\epsilon a}^{\epsilon b} \frac{f(t)-f(0)}{t}\,dt + f(0)\log\left(\frac{b}{a}\right).\end{aligned}$$

Since f is continuous at 0 and $t \geq \epsilon a$ in the interval of integration of length $\epsilon(b-a)$, we have

$$\left|\int_{\epsilon a}^{\epsilon b} \frac{f(t)-f(0)}{t}\,dt\right| \leq \left(\frac{b-a}{a}\right)\cdot \max\{|f(t)-f(0)| : \epsilon a \leq t \leq \epsilon b\} \to 0$$

as $\epsilon \to 0^+$. It follows that

$$I_1 = f(0)\log\left(\frac{b}{a}\right) - \int_a^b \frac{f(t)}{t}\,dt.$$

A similar argument gives

$$I_2 = -f(\infty)\log\left(\frac{b}{a}\right) + \int_a^b \frac{f(t)}{t}\,dt.$$

Adding these results, we conclude that

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}\,dx = \{f(0)-f(\infty)\}\log\left(\frac{b}{a}\right).$$

12.4.2. Absolutely convergent improper integrals. The convergence of improper integrals is analogous to the convergence of series. A series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges, and conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges. We introduce a similar definition for improper integrals and provide a test for the absolute convergence of an improper integral that is analogous to the comparison test for series.

Definition 12.32. An improper integral $\int_a^b f$ is absolutely convergent if the improper integral $\int_a^b |f|$ converges, and conditionally convergent if $\int_a^b f$ converges but $\int_a^b |f|$ diverges.

As part of the next theorem, we prove that an absolutely convergent improper integral converges (similarly, an absolutely convergent series converges).

Theorem 12.33. *Suppose that $f, g : I \to \mathbb{R}$ are defined on some finite or infinite interval I. If $|f| \leq g$ and the improper integral $\int_I g$ converges, then the improper integral $\int_I f$ converges absolutely. Moreover, an absolutely convergent improper integral converges.*

Proof. To be specific, we suppose that $f, g : [a, \infty) \to \mathbb{R}$ are integrable on $[a, r]$ for $r > a$ and consider the improper integral

$$\int_a^\infty f = \lim_{r\to\infty} \int_a^r f.$$

A similar argument applies to other types of improper integrals.

First, suppose that $f \geq 0$. Then

$$\int_a^r f \leq \int_a^r g \leq \int_a^\infty g,$$

so $\int_a^r f$ is a monotonic increasing function of r that is bounded from above. Therefore it converges as $r \to \infty$.

In general, we decompose f into its positive and negative parts,

$$\begin{aligned} f &= f_+ - f_-, & |f| &= f_+ + f_-, \\ f_+ &= \max\{f, 0\}, & f_- &= \max\{-f, 0\}. \end{aligned}$$

We have $0 \leq f_\pm \leq g$, so the improper integrals of $f_\pm$ converge by the previous argument, and therefore so does the improper integral of f:

$$\begin{aligned} \int_a^\infty f &= \lim_{r\to\infty} \left(\int_a^r f_+ - \int_a^r f_- \right) \\ &= \lim_{r\to\infty} \int_a^r f_+ - \lim_{r\to\infty} \int_a^r f_- \\ &= \int_a^\infty f_+ - \int_a^\infty f_-. \end{aligned}$$

Moreover, since $0 \leq f_\pm \leq |f|$, we see that $\int_a^\infty f_+$ and $\int_a^\infty f_-$ both converge if $\int_a^\infty |f|$ converges, and therefore so does $\int_a^\infty f$, so an absolutely convergent improper integral converges. $\square$

Example 12.34. Consider the limiting behavior of the error function $\operatorname{erf}(x)$ in Example 12.15 as $x \to \infty$, which is given by

$$\frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2}\, dx = \frac{2}{\sqrt{\pi}} \lim_{r\to\infty} \int_0^r e^{-x^2}\, dx.$$

The convergence of this improper integral follows by comparison with e^{-x}, for example, since

$$0 \leq e^{-x^2} \leq e^{-x} \qquad \text{for } x \geq 1,$$

and

$$\int_1^\infty e^{-x}\, dx = \lim_{r\to\infty} \int_1^r e^{-x}\, dx = \lim_{r\to\infty} \left(e^{-1} - e^{-r}\right) = \frac{1}{e}.$$

This argument proves that the error function approaches a finite limit as $x \to \infty$, but it doesn't give the exact value, only an upper bound

$$\frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2}\, dx \leq M, \qquad M = \frac{2}{\sqrt{\pi}} \left(\int_0^1 e^{-x^2}\, dx + \frac{1}{e} \right) \leq \frac{2}{\sqrt{\pi}} \left(1 + \frac{1}{e}\right).$$

One can evaluate this improper integral exactly, with the result that

$$\frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2}\, dx = 1.$$

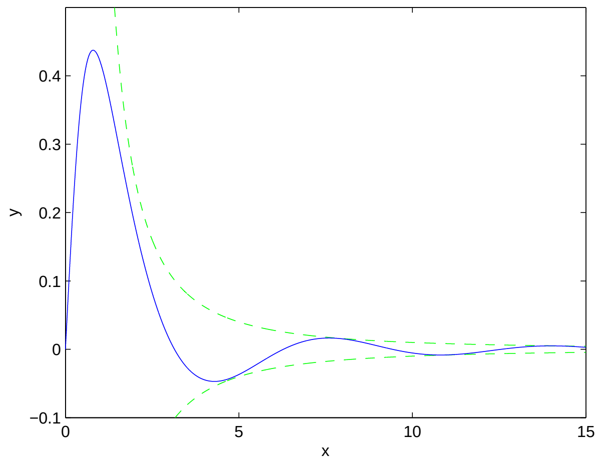

Figure 3. Graph of $y = (\sin x)/(1 + x^2)$ from Example 12.35. The dashed green lines are the graphs of $y = \pm 1/x^2$.

The standard trick to obtain this result (apparently introduced by Laplace) uses double integration, polar coordinates, and the substitution $u = r^2$:

$$\begin{aligned}\left(\int_0^\infty e^{-x^2}\,dx\right)^2 &= \int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dxdy \\ &= \int_0^{\pi/2} \left(\int_0^\infty e^{-r^2}\,r\,dr\right) d\theta \\ &= \frac{\pi}{4}\int_0^\infty e^{-u}\,du = \frac{\pi}{4}.\end{aligned}$$

This formal computation can be justified rigorously, but we won't do so here. There are also many other ways to obtain the same result.

Example 12.35. The improper integral

$$\int_0^\infty \frac{\sin x}{1+x^2}\,dx = \lim_{r\to\infty}\int_0^r \frac{\sin x}{1+x^2}\,dx$$

converges absolutely, since

$$\int_0^\infty \frac{\sin x}{1+x^2}\,dx = \int_0^1 \frac{\sin x}{1+x^2}\,dx + \int_1^\infty \frac{\sin x}{1+x^2}\,dx$$

and (see Figure 3)

$$\left|\frac{\sin x}{1+x^2}\right| \le \frac{1}{x^2} \quad \text{for } x \ge 1, \qquad \int_1^\infty \frac{1}{x^2}\,dx < \infty.$$

The value of this integral doesn't have an elementary expression, but by using contour integration from complex analysis one can show that

$$\int_0^\infty \frac{\sin x}{1+x^2}\,dx = \frac{1}{2e}\,\mathrm{Ei}(1) - \frac{e}{2}\,\mathrm{Ei}(-1) \approx 0.6468,$$

where Ei is the exponential integral function defined in Example 12.41.

Improper integrals, and the principal value integrals discussed below, arise frequently in complex analysis, and many such integrals can be evaluated by contour integration.

Example 12.36. The improper integral

$$\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{r\to\infty}\int_0^r \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

converges conditionally. We leave the proof as an exercise. Note that there is no difficulty at 0, since $\sin x/x \to 1$ as $x \to 0$, and comparison with the function $1/x$ doesn't imply absolute convergence at infinity because the improper integral $\int_1^\infty 1/x\,dx$ diverges. There are many ways to show that the exact value of the improper integral is $\pi/2$. The standard method uses contour integration.

Example 12.37. Consider the limiting behavior of the Fresnel sine function $S(x)$ in Example 12.16 as $x \to \infty$. The improper integral

$$\int_0^\infty \sin\left(\frac{\pi x^2}{2}\right)dx = \lim_{r\to\infty}\int_0^r \sin\left(\frac{\pi x^2}{2}\right)dx = \frac{1}{2}.$$

converges conditionally. This example may seem surprising since the integrand $\sin(\pi x^2/2)$ doesn't converge to 0 as $x \to \infty$. The explanation is that the integrand oscillates more rapidly with increasing x, leading to a more rapid cancelation between positive and negative values in the integral (see Figure 2). The exact value can be found by contour integration, again, which shows that

$$\int_0^\infty \sin\left(\frac{\pi x^2}{2}\right)dx = \frac{1}{\sqrt{2}}\int_0^\infty \exp\left(-\frac{\pi x^2}{2}\right)dx.$$

Evaluation of the resulting Gaussian integral gives $1/2$.

12.5. * Principal value integrals

Some integrals have a singularity that is too strong for them to converge as improper integrals but, due to cancelation between positive and negative parts of the integrand, they have a finite limit as a principal value integral. We begin with an example.

Example 12.38. Consider $f : [-1,1]\setminus\{0\}$ defined by

$$f(x) = \frac{1}{x}.$$

The definition of the integral of f on $[-1,1]$ as an improper integral is

$$\begin{aligned}\int_{-1}^1 \frac{1}{x}\,dx &= \lim_{\delta\to 0^+}\int_{-1}^{-\delta}\frac{1}{x}\,dx + \lim_{\epsilon\to 0^+}\int_\epsilon^1 \frac{1}{x}\,dx\\ &= \lim_{\delta\to 0^+}\log\delta - \lim_{\epsilon\to 0^+}\log\epsilon.\end{aligned}$$

Neither limit exists, so the improper integral diverges. (Formally, we get $\infty - \infty$.) If, however, we take $\delta = \epsilon$ and combine the limits, we get a convergent principal value integral, which is defined by

$$\text{p.v.} \int_{-1}^{1} \frac{1}{x}\, dx = \lim_{\epsilon \to 0^+} \left(\int_{-1}^{-\epsilon} \frac{1}{x}\, dx + \int_{\epsilon}^{1} \frac{1}{x}\, dx \right) = \lim_{\epsilon \to 0^+} (\log \epsilon - \log \epsilon) = 0.$$

The value of 0 is what one might expect from the oddness of the integrand. A cancelation in the contributions from either side of the singularity is essential to obtain a finite limit.

The principal value integral of $1/x$ on a non-symmetric interval about 0 still exists but is non-zero. For example, if $b > 0$, then

$$\text{p.v.} \int_{-1}^{b} \frac{1}{x}\, dx = \lim_{\epsilon \to 0^+} \left(\int_{-1}^{-\epsilon} \frac{1}{x}\, dx + \int_{\epsilon}^{b} \frac{1}{x}\, dx \right) = \lim_{\epsilon \to 0^+} (\log \epsilon + \log b - \log \epsilon) = \log b.$$

The crucial feature of a principal value integral is that we remove a symmetric interval around a singular point, or infinity. The resulting cancelation in the integral of a non-integrable function that changes sign across the singularity may lead to a finite limit.

Definition 12.39. If $f : [a, b] \setminus \{c\} \to \mathbb{R}$ is integrable on closed intervals not including $a < c < b$, then the principal value integral of f on $[a, b]$ is

$$\text{p.v.} \int_{a}^{b} f = \lim_{\epsilon \to 0^+} \left(\int_{a}^{c-\epsilon} f + \int_{c+\epsilon}^{b} f \right).$$

If $f : \mathbb{R} \to \mathbb{R}$ is integrable on compact intervals, then the principal value integral of f on $\mathbb{R}$ is

$$\text{p.v.} \int_{-\infty}^{\infty} f = \lim_{r \to \infty} \int_{-r}^{r} f.$$

If the improper integral exists, then the principal value integral exists and is equal to the improper integral. As Example 12.38 shows, the principal value integral may exist even if the improper integral does not. Of course, a principal value integral may also diverge.

Example 12.40. Consider the principal value integral

$$\begin{aligned} \text{p.v.} \int_{-1}^{1} \frac{1}{x^2}\, dx &= \lim_{\epsilon \to 0^+} \left(\int_{-1}^{-\epsilon} \frac{1}{x^2}\, dx + \int_{\epsilon}^{1} \frac{1}{x^2}\, dx \right) \\ &= \lim_{\epsilon \to 0^+} \left(\frac{2}{\epsilon} - 2 \right) = \infty. \end{aligned}$$

In this case, the function $1/x^2$ is positive and approaches ∞ on both sides of the singularity at $x = 0$, so there is no cancelation and the principal value integral diverges to ∞.

Principal value integrals arise frequently in complex analysis, harmonic analysis, and a variety of applications.

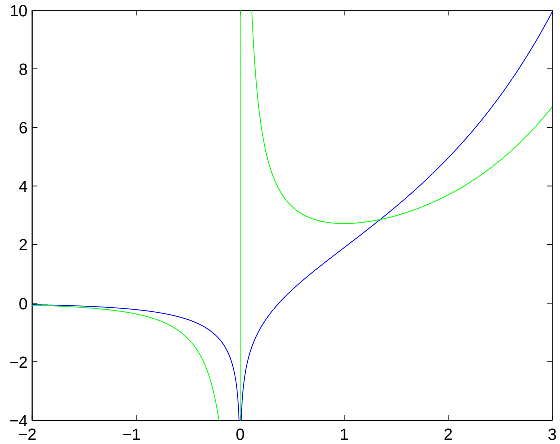

Figure 4. Graphs of the exponential integral $y = \mathrm{Ei}(x)$ (blue) and its derivative $y = e^x/x$ (green) from Example 12.41.

Example 12.41. The exponential integral Ei is a non-elementary function defined by

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t}\,dt.$$

Its graph is shown in Figure 4. This integral has to be understood, in general, as an improper, principal value integral, and the function has a logarithmic singularity at $x = 0$.

If $x < 0$, then the integrand is continuous for $-\infty < t \le x$, and the integral is interpreted as an improper integral,

$$\int_{\infty}^{x} \frac{e^t}{t}\,dt = \lim_{r\to\infty} \int_{-r}^{x} \frac{e^t}{t}\,dt.$$

This improper integral converges absolutely by comparison with e^t, since

$$\left|\frac{e^t}{t}\right| \le e^t \qquad \text{for } -\infty < t \le -1,$$

and

$$\int_{-\infty}^{-1} e^t\,dt = \lim_{r\to\infty} \int_{-r}^{-1} e^t\,dt = \lim_{r\to\infty} \left(e^{-r} - e^{-1}\right) = \frac{1}{e}.$$

If $x > 0$, then the integrand has a non-integrable singularity at $t = 0$, and we interpret it as a principal value integral. We write

$$\int_{-\infty}^{x} \frac{e^t}{t}\,dt = \int_{-\infty}^{-1} \frac{e^t}{t}\,dt + \int_{-1}^{x} \frac{e^t}{t}\,dt.$$

The first integral is interpreted as an improper integral as before. The second integral is interpreted as a principal value integral

$$\text{p.v.}\int_{-1}^{x} \frac{e^t}{t}\,dt = \lim_{\epsilon\to 0^+}\left(\int_{-1}^{-\epsilon} \frac{e^t}{t}\,dt + \int_{\epsilon}^{x} \frac{e^t}{t}\,dt\right).$$

This principal value integral converges, since

$$\text{p.v.}\int_{-1}^{x} \frac{e^t}{t}\,dt = \int_{-1}^{x} \frac{e^t-1}{t}\,dt + \text{p.v.}\int_{-1}^{x} \frac{1}{t}\,dt = \int_{-1}^{x} \frac{e^t-1}{t}\,dt + \log x.$$

The first integral makes sense as a Riemann integral since the integrand has a removable singularity at $t = 0$, with

$$\lim_{t\to 0}\left(\frac{e^t-1}{t}\right) = 1,$$

so it extends to a continuous function on $[-1, x]$.

Finally, if $x = 0$, then the integrand is unbounded at the left endpoint $t = 0$. The corresponding improper integral diverges, and $\mathrm{Ei}(0)$ is undefined.

The exponential integral arises in physical applications such as heat flow and radiative transfer. It is also related to the logarithmic integral

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\log t}$$

by $\mathrm{li}(x) = \mathrm{Ei}(\log x)$. The logarithmic integral is important in number theory, where it gives an asymptotic approximation for the number of primes less than x as $x \to \infty$.

Example 12.42. Let $f : \mathbb{R} \to \mathbb{R}$ and assume, for simplicity, that f has compact support, meaning that $f = 0$ outside a compact interval $[-r, r]$. If f is integrable, we define the Hilbert transform $Hf : \mathbb{R} \to \mathbb{R}$ of f by the principal value integral

$$Hf(x) = \frac{1}{\pi}\text{p.v.}\int_{-\infty}^{\infty} \frac{f(t)}{x-t}\,dt = \frac{1}{\pi}\lim_{\epsilon\to 0^+}\left(\int_{-\infty}^{x-\epsilon} \frac{f(t)}{x-t}\,dt + \int_{x+\epsilon}^{\infty} \frac{f(t)}{x-t}\,dt\right).$$

Here, x plays the role of a parameter in the integral with respect to t. We use a principal value because the integrand may have a non-integrable singularity at $t = x$. Since f has compact support, the intervals of integration are bounded and there is no issue with the convergence of the integrals at infinity.

For example, suppose that f is the step function

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1, \\ 0 & \text{for } x < 0 \text{ or } x > 1. \end{cases}$$

If $x < 0$ or $x > 1$, then $t \ne x$ for $0 \le t \le 1$, and we get a proper Riemann integral

$$Hf(x) = \frac{1}{\pi}\int_0^1 \frac{1}{x-t}\,dt = \frac{1}{\pi}\log\left|\frac{x}{x-1}\right|.$$

If $0 < x < 1$, then we get a principal value integral

$$\begin{aligned} Hf(x) &= \frac{1}{\pi} \lim_{\epsilon \to 0^+} \left(\int_0^{x-\epsilon} \frac{1}{x-t}\, dt + \frac{1}{\pi} \int_{x+\epsilon}^1 \frac{1}{x-t}\, dt \right) \\ &= \frac{1}{\pi} \lim_{\epsilon \to 0^+} \left[\log\left(\frac{x}{\epsilon}\right) + \log\left(\frac{\epsilon}{1-x}\right) \right] \\ &= \frac{1}{\pi} \log\left(\frac{x}{1-x}\right) \end{aligned}$$

Thus, for $x \neq 0, 1$ we have

$$Hf(x) = \frac{1}{\pi} \log \left| \frac{x}{x-1} \right|.$$

The principal value integral with respect to t diverges if $x = 0, 1$ because $f(t)$ has a jump discontinuity at the point where $t = x$. Consequently the values $Hf(0)$, $Hf(1)$ of the Hilbert transform of the step function are undefined.

12.6. The integral test for series

An a further application of the improper integral, we prove a useful test for the convergence or divergence of a monotone decreasing, positive series. The idea is to interpret the series as an upper or lower sum of an integral.

Theorem 12.43 (Integral test)**.** Suppose that $f : [1, \infty) \to \mathbb{R}$ is a positive decreasing function (i.e., $0 \leq f(x) \leq f(y)$ for $x \geq y$). Let $a_n = f(n)$. Then the series

$$\sum_{n=1}^{\infty} a_n$$

converges if and only if the improper integral

$$\int_1^{\infty} f(x)\, dx$$

converges. Furthermore, the limit

$$D = \lim_{n \to \infty} \left[\sum_{k=1}^{n} a_k - \int_1^n f(x)\, dx \right]$$

exists, and $0 \leq D \leq a_1$.

Proof. Let

$$S_n = \sum_{k=1}^{n} a_k, \qquad T_n = \int_1^n f(x)\, dx.$$

The integral T_n exists since f is monotone, and the sequences (S_n), (T_n) are increasing since f is positive.

Let

$$P_n = \{[1, 2], [2, 3], \dots, [n-1, n]\}$$

be the partition of $[1, n]$ into $n - 1$ intervals of length 1. Since f is decreasing,

$$\sup_{[k,k+1]} f = a_k, \qquad \inf_{[k,k+1]} f = a_{k+1},$$

and the upper and lower sums of f on P_n are given by

$$U(f; P_n) = \sum_{k=1}^{n-1} a_k, \qquad L(f; P_n) = \sum_{k=1}^{n-1} a_{k+1}.$$

Since the integral of f on $[1, n]$ is bounded by its upper and lower sums, we get that

$$S_n - a_1 \le T_n \le S_{n-1}.$$

This inequality shows that (T_n) is bounded from above by S if $S_n \uparrow S$, and (S_n) is bounded from above by $T + a_1$ if $T_n \uparrow T$, so (S_n) converges if and only if (T_n) converges, which proves the first part of the theorem.

Let $D_n = S_n - T_n$. Then the inequality shows that $a_n \le D_n \le a_1$; in particular, (D_n) is bounded from below by zero. Moreover, since f is decreasing,

$$D_n - D_{n+1} = \int_n^{n+1} f(x)\, dx - a_{n+1} \ge f(n+1) \cdot 1 - a_{n+1} = 0,$$

so (D_n) is decreasing. Therefore $D_n \downarrow D$ where $0 \le D \le a_1$, which proves the second part of the theorem. $\square$

A basic application of this result is to the p-series.

Example 12.44. Applying Theorem 12.43 to the function $f(x) = 1/x^p$ and using Example 12.27, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if $p > 1$ and diverges if $0 < p \le 1$.

Theorem 12.43 is also useful for divergent series, since it tells us how quickly their partial sums diverge. We remark that one can obtain similar, but more accurate, asymptotic approximations than the one in theorem for the behavior of the partial sums in terms of integrals, called the Euler-MacLaurin summation formulae.

Example 12.45. Applying the second part of Theorem 12.43 to the function $f(x) = 1/x$, we find that

$$\lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \log n\right] = \gamma$$

where the limit $0 \le \gamma < 1$ is the Euler constant.

Example 12.46. We can use the result of Example 12.45 to compute the sum A of the alternating harmonic series

$$A = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots.$$

The partial sum of the first $2m$ terms is given by

$$\begin{aligned}
A_{2m} &= \sum_{k=1}^{m} \frac{1}{2k-1} - \sum_{k=1}^{m} \frac{1}{2k} \\
&= \sum_{k=1}^{2m} \frac{1}{k} - 2\sum_{k=1}^{m} \frac{1}{2k} \\
&= \sum_{k=1}^{2m} \frac{1}{k} - \sum_{k=1}^{m} \frac{1}{k}.
\end{aligned}$$

Here, we rewrite a sum of the odd terms in the harmonic series as the difference between the harmonic series and its even terms, then use the fact that a sum of the even terms in the harmonic series is one-half the sum of the series. It follows that

$$\lim_{m\to\infty} A_{2m} = \lim_{m\to\infty} \left\{ \sum_{k=1}^{2m} \frac{1}{k} - \log 2m - \left[\sum_{k=1}^{m} \frac{1}{k} - \log m \right] + \log 2m - \log m \right\}.$$

Since $\log 2m - \log m = \log 2$, we get that

$$\lim_{m\to\infty} A_{2m} = \gamma - \gamma + \log 2 = \log 2.$$

The odd partial sums also converge to $\log 2$ since

$$A_{2m+1} = A_{2m} + \frac{1}{2m+1} \to \log 2 \qquad \text{as } m \to \infty.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log 2.$$

Example 12.47. A similar calculation to the previous one can be used to to compute the sum S of the rearrangement of the alternating harmonic series

$$S = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \dots$$

discussed in Example 4.32. The partial sum S_{3m} of the series may be written in terms of partial sums of the harmonic series as

$$\begin{aligned}
S_{3m} &= 1 - \frac{1}{2} - \frac{1}{4} + \dots + \frac{1}{2m-1} - \frac{1}{4m-2} - \frac{1}{4m} \\
&= \sum_{k=1}^{m} \frac{1}{2k-1} - \sum_{k=1}^{2m} \frac{1}{2k} \\
&= \sum_{k=1}^{2m} \frac{1}{k} - \sum_{k=1}^{m} \frac{1}{2k} - \sum_{k=1}^{2m} \frac{1}{2k} \\
&= \sum_{k=1}^{2m} \frac{1}{k} - \frac{1}{2}\sum_{k=1}^{m} \frac{1}{k} - \frac{1}{2}\sum_{k=1}^{2m} \frac{1}{k}.
\end{aligned}$$

It follows that

$$\lim_{m\to\infty} S_{3m} = \lim_{m\to\infty} \left\{ \sum_{k=1}^{2m} \frac{1}{k} - \log 2m - \frac{1}{2}\left[\sum_{k=1}^{m} \frac{1}{k} - \log m\right] - \frac{1}{2}\left[\sum_{k=1}^{2m} \frac{1}{k} - \log 2m\right] \right.$$
$$\left. + \log 2m - \frac{1}{2}\log - \frac{1}{2}\log 2m \right\}.$$

Since $\log 2m - \frac{1}{2}\log m - \frac{1}{2}\log 2m = \frac{1}{2}\log 2$, we get that that

$$\lim_{m\to\infty} S_{3m} = \gamma - \frac{1}{2}\gamma - \frac{1}{2}\gamma + \frac{1}{2}\log 2 = \frac{1}{2}\log 2.$$

Finally, since

$$\lim_{m\to\infty} (S_{3m+1} - S_{3m}) = \lim_{m\to\infty} (S_{3m+2} - S_{3m}) = 0,$$

we conclude that the whole series converges to $S = \frac{1}{2}\log 2$.

12.7. Taylor's theorem with integral remainder

In Theorem 8.46, we gave an expression for the error between a function and its Taylor polynomial of order n in terms of the Lagrange remainder, which involves a pointwise value of the derivative of order $n+1$ evaluated at some intermediate point. In the next theorem, we give an alternative expression for the error in terms of an integral of the derivative.

Theorem 12.48 (Taylor with integral remainder)**.** Suppose that $f : (a,b) \to \mathbb{R}$ has $n+1$ derivatives on (a,b) and f^{n+1} is Riemann integrable on every subinterval of (a,b). Let $a < c < b$. Then for every $a < x < b$,

$$f(x) = f(c) + f'(c)(x-c) + \frac{1}{2!}f''(c)(x-c)^2 + \cdots + \frac{1}{n!}f^{(n)}(c)(x-c)^n + R_n(x)$$

where

$$R_n(x) = \frac{1}{n!}\int_c^x f^{(n+1)}(t)(x-t)^n\,dt.$$

Proof. We use proof by induction. The formula is true for $n = 0$, since the fundamental theorem of calculus (Theorem 12.1) implies that

$$f(x) = f(c) + \int_c^x f'(t)\,dt = f(c) + R_0(x).$$

Assume that the formula is true for some $n \in \mathbb{N}_0$ and f^{n+2} is Riemann integrable. Then, since

$$(x-t)^n = -\frac{1}{n+1}\frac{d}{dt}(x-t)^{n+1},$$

an integration by parts with respect to t (Theorem 12.10) implies that

$$\begin{aligned} R_n(x) &= -\left[\frac{1}{(n+1)!}f^{(n+1)}(t)(x-t)^{n+1}\right]_c^x + \frac{1}{(n+1)!}\int_c^x f^{(n+2)}(t)(x-t)^{n+1}\,dt \\ &= \frac{1}{(n+1)!}f^{(n+1)}(c)(x-c)^{n+1} + R_{n+1}(x). \end{aligned}$$

Use of this equation in the formula for n gives the formula for $n+1$, which proves the result. $\square$

By making the change of variable

$$t = c + s(x - c),$$

we can also write the remainder as

$$R_n(x) = \frac{1}{n!}\left[\int_0^1 f^{(n+1)}\left(c + s(x-c)\right)(1-s)^n\, ds\right](x-c)^{n+1}.$$

In particular, if $|f^{(n+1)}(x)| \le M$ for $a < x < b$, then

$$\begin{aligned} |R_n(x)| &\le \frac{1}{n!} M \left[\int_0^1 (1-s)^n\, ds\right]|x-c|^{n+1} \\ &\le \frac{M}{(n+1)!}|x-c|^{n+1}, \end{aligned}$$

which agrees with what one gets from the Lagrange remainder.

Thus, the integral form of the remainder is as effective as the Lagrange form in estimating its size from a uniform bound on the derivative. The integral form requires slightly stronger assumptions than the Lagrange form, since we need to assume that the derivative of order $n+1$ is integrable, but its proof is straightforward once we have the integral. Moreover, the integral form generalizes to vector-valued functions $f : (a, b) \to \mathbb{R}^n$, while the Lagrange form does not.