

An Introduction to Real Analysis

John K. Hunter

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT DAVIS

ABSTRACT. These are some notes on introductory real analysis. They cover the properties of the real numbers, sequences and series of real numbers, limits of functions, continuity, differentiability, sequences and series of functions, and Riemann integration. They don't include multi-variable calculus or contain any problem sets. Optional sections are starred.

Contents

Chapter 1. Sets and Functions	1
1.1. Sets	1
1.2. Functions	5
1.3. Composition and inverses of functions	7
1.4. Indexed sets	8
1.5. Relations	11
1.6. Countable and uncountable sets	14
Chapter 2. Numbers	21
2.1. Integers	22
2.2. Rational numbers	23
2.3. Real numbers: algebraic properties	25
2.4. Real numbers: ordering properties	26
2.5. The supremum and infimum	27
2.6. Real numbers: completeness	29
2.7. Properties of the supremum and infimum	31
2.8. Density of the Rationals	35
Chapter 3. Sequences	37
3.1. The absolute value	37
3.2. Sequences	38
3.3. Convergence and limits	41
3.4. Properties of limits	45
3.5. Monotone sequences	47
3.6. The lim sup and lim inf	50
3.7. Cauchy sequences	56

3.8. Subsequences	57
3.9. The Bolzano-Weierstrass theorem	59
Chapter 4. Series	61
4.1. Convergence of series	61
4.2. The Cauchy condition	64
4.3. Absolutely convergent series	66
4.4. The comparison test	68
4.5. * The Riemann ζ -function	70
4.6. The ratio and root tests	71
4.7. Alternating series	73
4.8. Rearrangements	75
4.9. The Cauchy product	79
4.10. * Double series	80
4.11. * The irrationality of e	88
Chapter 5. Topology of the Real Numbers	91
5.1. Open sets	91
5.2. Closed sets	94
5.3. Compact sets	97
5.4. Connected sets	104
5.5. * The Cantor set	106
Chapter 6. Limits of Functions	111
6.1. Limits	111
6.2. Left, right, and infinite limits	116
6.3. Properties of limits	119
Chapter 7. Continuous Functions	123
7.1. Continuity	123
7.2. Properties of continuous functions	127
7.3. Uniform continuity	129
7.4. Continuous functions and open sets	131
7.5. Continuous functions on compact sets	133
7.6. The intermediate value theorem	135
7.7. Monotonic functions	138
Chapter 8. Differentiable Functions	141
8.1. The derivative	141
8.2. Properties of the derivative	147
8.3. The chain rule	149
8.4. Extreme values	152

8.5.	The mean value theorem	154
8.6.	Taylor's theorem	156
8.7.	* The inverse function theorem	159
8.8.	* L'Hôpital's rule	164
Chapter 9.	Sequences and Series of Functions	169
9.1.	Pointwise convergence	169
9.2.	Uniform convergence	171
9.3.	Cauchy condition for uniform convergence	172
9.4.	Properties of uniform convergence	173
9.5.	Series	177
Chapter 10.	Power Series	183
10.1.	Introduction	183
10.2.	Radius of convergence	184
10.3.	Examples of power series	186
10.4.	Algebraic operations on power series	190
10.5.	Differentiation of power series	195
10.6.	The exponential function	197
10.7.	* Smooth versus analytic functions	199
Chapter 11.	The Riemann Integral	207
11.1.	The supremum and infimum of functions	208
11.2.	Definition of the integral	210
11.3.	The Cauchy criterion for integrability	217
11.4.	Continuous and monotonic functions	221
11.5.	Linearity, monotonicity, and additivity	224
11.6.	Further existence results	232
11.7.	* Riemann sums	236
11.8.	* The Lebesgue criterion	240
Chapter 12.	Properties and Applications of the Integral	243
12.1.	The fundamental theorem of calculus	243
12.2.	Consequences of the fundamental theorem	248
12.3.	Integrals and sequences of functions	253
12.4.	Improper Riemann integrals	257
12.5.	* Principal value integrals	263
12.6.	The integral test for series	267
12.7.	Taylor's theorem with integral remainder	270
Chapter 13.	Metric, Normed, and Topological Spaces	273
13.1.	Metric spaces	273

13.2. Normed spaces	278
13.3. Open and closed sets	281
13.4. Completeness, compactness, and continuity	284
13.5. Topological spaces	289
13.6. * Function spaces	291
13.7. * The Minkowski inequality	295
Bibliography	301

