Math 118 Fall 2013

Homework 2 Solutions

1.3.1

- Applying Newton's Law $F = ma$ in both x- and y-directions. Refer to class notes for assumptions.
- For x-direction, no motion implies no resistance force. We still have as in class

$$\begin{aligned} & T(b,t)\cos(\theta(b,t)) - T(a,t)\cos(\theta(a,t)) = 0 \\ \implies & T(b,t) - T(a,t) = 0 \\ \implies & T = T_0 = \text{constant}. \end{aligned}$$

- For y-direction we have

$$\begin{aligned} T(b,t)\sin(\theta(b,t)) - T(a,t)\sin(\theta(a,t)) - \int_a^b \alpha u_t \, dx &= \int_a^b \rho_0 u_{tt} \, dx \\ T_0\left(u_x(b,t) - u_x(a,t)\right) - \int_a^b \alpha u_t \, dx &= \int_a^b \rho_0 u_{tt} \, dx \\ \int_a^b T_0 u_{xx}(x,t) \, dx - \int_a^b \alpha u_t \, dx &= \int_a^b \rho_0 u_{tt} \, dx \\ \int_a^b T_0 u_{xx}(x,t) - \alpha u_t - \rho_0 u_{tt} \, dx &= 0, \end{aligned}$$

  for arbitrary points a, b. Thus

$$T_0 u_{xx}(x,t) - \alpha u_t - \rho_0 u_{tt} = 0,$$

  or

$$u_{tt} - c^2 u_{xx} + r u_t = 0,$$

  where $c^2 = \frac{T_0}{\rho_0}, r = \frac{\alpha}{\rho_0} > 0$.
- Resistance force is again the motion, so $r > 0$.

1.5.1

- Characteristic equation is $r^2 + 1 = 0$ which has roots $r = \pm i$.
- General solution has the form $u(x) = A\cos x + B\sin x$.
- Applying boundary condition $u(0) = 0$ leads to $A = 0$, and $u(x) = B\sin x$.
- Another boundary condition $u(L) = 0$ implies $B\sin L = 0$. There are two possibilities:
  - if $L \neq k\pi$ for $k \in \mathbb{Z}$, B has to be zero, and $u(x) = 0$ is the unique solution.
  - if $L = k\pi$ for $k \in \mathbb{Z}$, then $u(x) = B\sin L$ are all solutions.

2.1.1

- The general solution is $u(x,t) = f(x-ct) + g(x-ct)$.
- Applying the initial conditions lead to $f(x) + g(x) = e^x$, and $-cf'(x) + cg'(x) = \sin x$.
- Solving for f and g, we obtain

$$f(x) = \frac{1}{2}(e^x + \frac{1}{c}\cos x), \quad g(x) = \frac{1}{2}(e^x - \frac{1}{c}\cos x).$$

- Solution of the IVP is

$$\begin{aligned} u(x,t) &= \frac{1}{2}e^x(e^{-ct} + c^{ct}) + \frac{1}{2c}(\cos(x-ct) - \cos(x+ct)) \\ &= e^x\cosh(ct) + \frac{1}{c}\sin(x)\sin(ct). \end{aligned}$$

2.1.5

- At $t = a/2c$,

$$u(x, a/2c) = \begin{cases} 0 \text{ if } |x| > \frac{3a}{2} \\ \frac{1}{2c}(\frac{3a}{2} - |x|) \text{ if } \frac{a}{2} \le |x| \le \frac{3a}{2} \\ \frac{a}{2c} \text{ if } |x| < \frac{a}{2} \end{cases}$$

- At $t = a/c$,

$$u(x, a/c) = \begin{cases} 0 \text{ if } |x| > 2a \\ \frac{1}{2c}(2a - |x|) \text{ if } |x| \le 2a \end{cases}$$

- At $t = 3a/2c$,

$$u(x, 3a/2c) = \begin{cases} 0 \text{ if } |x| > \frac{5a}{2} \\ \frac{1}{2c}(\frac{5a}{2} - |x|) \text{ if } \frac{a}{2} \le |x| \le \frac{5a}{2} \\ \frac{a}{c} \text{ if } |x| < \frac{a}{2} \end{cases}$$

- At $t = 2a/c$,

$$u(x, 2a/c) = \begin{cases} 0 \text{ if } |x| > 3a \\ \frac{1}{2c}(3a - |x|) \text{ if } a \le |x| \le 3a \\ \frac{a}{c} \text{ if } |x| < a \end{cases}$$

- At $t = 5a/c$,

$$u(x, 5a/c) = \begin{cases} 0 \text{ if } |x| > 6a \\ \frac{1}{2c}(6a - |x|) \text{ if } 4a \le |x| \le 6a \\ \frac{a}{c} \text{ if } |x| < 4a \end{cases}$$

- Note the pattern. The graph changes slopes at points $x = \pm(a \pm ct)$. Except the horizontal lines, all slopes have magnitude $\frac{1}{2c}$.

2.1.6

$$\max_x u(x,t) = \begin{cases} a/c \text{ for } t \geq a/c \\ t \text{ for } 0 \leq t < a/c \end{cases}$$

2.2.2

- For part (a), differentiating e and p with respect to x and t, respectively. Then use the wave equation $u_{tt} - u_{xx} = 0$ to show.
- For part (b), use the assumptions that the second-partial derivatives are continuous, i.e, $e_{xt} = e_{tx}$, $p_{xt} = p_{tx}$.