Math 118: PDE

HW 3 Solutions

1.3.3

- By the law of conservation of energy, we have

$$\text{rate of change of thermal energy} = \text{heat flux in} - \text{heat flux out}. \quad (1)$$

- Let $u(x,t)$ denote the temperature of the rod at postition x and time t. The thermal energy density $e(x,t)$ per volume is given by $c\rho u(x,t)$, where c is the specific heat capacity of the rod, and ρ the density.

- The heat flux $q(x,t)$ per cross sectional area is proportional to the negative gradient of temperature, i.e. $q(x,t) = -ku_x$ for some $k > 0$.

- The addtional heat loss to the outside through the lateral sides of the rod is given by $h(x,t)P\delta x$ where $h(x,t) = \mu[u(x,t) - T_0]$ for some $\mu > 0$, where T_0 is the ambient temperature, based on Newton's law of cooling.

- By (1), we have

$$\frac{d}{dt}\int_a^b e(x,t)A\,dx = [q(a,t)A - q(b,t)A] - \int_a^b h(x,t)P\,dx, \quad (2)$$

  which can be simplified to

$$\int_a^b c\rho A u_t - Aku_{xx} + \mu P[u - T_0]\,dx = 0. \quad (3)$$

- Since a, b are arbitrary, it follows that

$$u_t = \frac{k}{c\rho}u_{xx} - \frac{\mu P}{c\rho A}[u - T_0]. \quad (4)$$

1.3.5

- By the conservation of mass, we have

$$\begin{aligned} &\text{the rate of change in the fluid mass} \\ &= \text{change in flux based on diffusion (i.e.,flow in - flow out)} \quad (5) \\ &+ \text{change in flux based on advection (i.e., move in - move out).} \end{aligned}$$

- Let $u(x, t)$ denote the mass of fluid particles at position x and time t. Then (5) gives

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = -\int_a^b q_x(x,t)\,dx - \int_a^b Vu_x(x,t)\,dx, \tag{6}$$

- Since $q = -ku_t$ and a, b are chosen arbitrarily, we have

$$u_t = ku_{xx} - Vu_x. \tag{7}$$

2.2.6

- (a) Substitude the following derivative to the PDE

$$\begin{aligned}
u_t &= \alpha f'(t-\beta)\\
u_{tt} &= \alpha f''(t-\beta)\\
u_r &= \alpha' f(t-\beta) - \alpha\beta' f'(t-\beta)\\
u_{rr} &= \alpha'' f(t-\beta) - 2\alpha'\beta' f'(t-\beta) - \alpha\beta'' f'(t-\beta) + \alpha(\beta')^2 f''(t-\beta)
\end{aligned}$$

we get

$$c^2(\alpha''+\frac{n-1}{r}\alpha')f - c^2(2\alpha'\beta'+\alpha\beta''+\frac{n-1}{r}\alpha\beta')f' + (c^2\alpha(\beta')^2-\alpha)f'' = 0 \tag{8}$$

- (b) Setting the coefficients of f'', f',and f equal to zero, we obtain

$$c^2(\alpha'' + \frac{n-1}{r}\alpha') = 0 \tag{9}$$

$$c^2(2\alpha'\beta' + \alpha\beta'' + \frac{n-1}{r}\alpha\beta') = 0 \tag{10}$$

$$c^2\alpha(\beta')^2 - \alpha = 0 \tag{11}$$

- (c) Suppose that $\alpha \neq 0$ and $c \neq 0$, then (11) gives $\beta' = \pm 1/c$ and thus $\beta'' = 0$. Plug these results to (20), we obtain

$$2\alpha' + \frac{n-1}{r}\alpha = 0 \tag{12}$$

- The equation (9) gives

$$\alpha'' + \frac{n-1}{r}\alpha' = 0. \tag{13}$$

Solving this ODE, we obtain

$$\alpha' = r^{1-n}, \quad \alpha = \frac{1}{2-n}r^{2-n}$$

- Plugging them into (12), we get

$$r^{1-n}\left(2+\frac{n-1}{2-n}\right)=0. \tag{14}$$

  It follows that $n=1$ or $n=3$.

- (d) If $n=1$, $\alpha(r)=r^0=1$ is a constant.

2.3.1

- Maximum Principle tells that the max or min of $u(x,t)$ occurs on the boundaries, i.e., $t=0,T$, $x=0,1$.

	function	max	min
$t=0$	$u(x,0)=1-x^2$	1 at $x=0$	0 at $x=1$
$x=0$	$u(0,t)=1-2kt$	1 at $t=0$	$1-2kT$ at $t=T$
$x=1$	$u(1,t)=-2kt$	0 at $t=0$	$-2kT$ at $t=T$

- Thus, the global max of $u(x,t)$ is 1 at $(0,0)$, and the global min is $-2kT$ at $(1,T)$.

2.4.6

- Let $I=\int_0^\infty e^{-x^2}\,dx$, then

$$\begin{aligned} I^2 &= \int_0^\infty e^{-x^2}\,dx \int_0^\infty e^{-y^2}\,dy \\ &= \int_0^\infty \int_0^\infty e^{-x^2}e^{-y^2}\,dx\,dy \\ &= \int_0^{\pi/4} \int_0^\infty e^{-r^2} r\,dr\,d\theta \\ &= \pi/4. \end{aligned}$$

  Thus, $I=\sqrt{\pi}/2$.

2.4.7

- $\int_{-\infty}^\infty e^{-p^2}\,dp = 2I = \sqrt{\pi}$.

- Let $p=x/\sqrt{4kt}$, then $dp=dx/\sqrt{4kt}$, and

$$\int_{-\infty}^\infty S(x,t)\,dx = \int_{-\infty}^\infty \frac{e^{-p^2}}{\sqrt{\pi}}\,dp = 1.$$

2.4.9

- Differentiating both sides of the diffusion equation thrice with respect to x, we have
$$(u_{xxx})_t = k(u_{xxx})_{xx}, \tag{15}$$
due to the continuity of partial derivatives.

- Differentiating $u(x,0) = x^2$ thrice with respect to x, we have the initial condition
$$u_{xxx}(x,0) = 0. \tag{16}$$

- By the uniqueness of solutions, $u_{xxx} = 0$ is the solution of the IVP (15) and (16).

- Integrating the result thrice,
$$u(x,t) = A(t)x^2 + B(t)x + C(t). \tag{17}$$

- The initial condition $u(x,0) = x^2$ implies
$$A(0) = 1, \quad B(0) = C(0) = 0.$$

- Differentiating (17) with respect to t,
$$u_t = A'x^2 + B'x + C' \tag{18}$$

- Differentiating (17) with respect to x twice,
$$u_{xx} = 2A. \tag{19}$$

- Plugging (18) and (19) into the orginal diffusion equation, we obtain
$$A'(t) = B'(t) = 0, \quad C'(t) = 2kA(t). \tag{20}$$

- It follows that $A = 1,\ B = 0, C = 2k$, and the solution of the original problem is $u(x,t) = x^2 + 2kt$.