Math 118: PDE

HW 4 Solutions

2.4.1

- The solution of the IVP is

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\,dy$$
$$= \frac{1}{\sqrt{4\pi kt}} \int_{-l}^{l} e^{-(x-y)^2/4kt}\,dy.$$

- Substitute $p = (x-y)/\sqrt{4kt}$ in this integral, we obtain

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{\frac{x-l}{\sqrt{4kt}}}^{\frac{x+l}{\sqrt{4kt}}} e^{-p^2}\,dp$$
$$= \frac{1}{\sqrt{\pi}} \left(\int_0^{\frac{x+l}{\sqrt{4kt}}} - \int_0^{\frac{x-l}{\sqrt{4kt}}} \right) e^{-p^2}\,dp$$
$$= \frac{1}{2}\left[\mathrm{Erf}\left(\frac{x+l}{\sqrt{4kt}}\right) - \mathrm{Erf}\left(\frac{x-l}{\sqrt{4kt}}\right) \right].$$

2.4.10

- The solution is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\,dy$$
$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}y^2\,dy.$$

- Substitute $p = (x-y)/\sqrt{4kt}$ in this integral, we obtain

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2}(x - \sqrt{4kt}p)^2\,dp$$
$$= x^2 - \frac{2\sqrt{4kt}x}{\sqrt{\pi}} \int_{-\infty}^{\infty} pe^{-p^2}\,dp + \frac{4kt}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^2 e^{-p^2}\,dp$$
$$= x^2 + \frac{4kt}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^2 e^{-p^2}\,dp.$$

- The integral
$$\int_{-\infty}^{\infty} pe^{-p^2}\, dp$$
is zero due to its odd integrand.

- From Exercise 9, the solution is $u(x,t) = x^2 + 2kt$. By the uniqueness of solution, we must have
$$\frac{4kt}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^2 e^{-p^2}\, dp = 2kt,$$
which leads to
$$\int_{-\infty}^{\infty} p^2 e^{-p^2}\, dp = \frac{\sqrt{\pi}}{2}.$$

2.5 2

- Given $u(x,t) = f(x - at)$, then
$$u_{tt} = a^2 f''(x - at), \quad u_{xx} = f''(x - at).$$

- If u is a solution of the wave equation, then $u_{tt} = c^2 u_{xx}$, and
$$a^2 f''(x - at) = c^2 f''(x - at),$$
which gives $a = \pm c$.

- If u is a solution of the diffusion equation, then $u_t = k u_{xx}$, and
$$-af'(x - at) = kf''(x - at),$$
which can be written as a second order ODE
$$f'' + \frac{a}{k} f' = 0.$$

- The general solution of the ODE is $f(x) = A + Be^{-ax/k}$, where A, B are arbitrary constants.

- We thus have
$$u(x,t) = A + Be^{-a(x-at)/k}.$$

2.5.3

- Find v_t using the product and chain rules:

$$\begin{aligned} v_t &= -\frac{1}{2t^{3/2}}e^{x^2/2t}u - \frac{x^2}{2t^{5/2}}e^{x^2/2t}u - \frac{x}{t^{5/2}}e^{x^2/2t}u_x - \frac{1}{t^{5/2}}e^{x^2/2t}u_t \\ &= -\frac{1}{2t^{5/2}}e^{x^2/2t}\left(tu + x^2u + 2xu_x + 2u_t\right) \\ &= -\frac{1}{2t^{5/2}}e^{x^2/2t}\left(tu + x^2u + 2xu_x + u_{xx}\right), \end{aligned}$$

  because $u_t = \frac{1}{2}u_{xx}$.

- Similarly,

$$v_x = \frac{1}{\sqrt{t}}e^{x^2/2t}\left(\frac{x}{t}u + \frac{1}{t}u_x\right),$$

  and

$$\begin{aligned} v_{xx} &= \frac{1}{\sqrt{t}}e^{x^2/2t}\left(\frac{x}{t}(\frac{x}{t}u + \frac{1}{t}u_x) + (\frac{1}{t}u + \frac{x}{t^2}u_x) + \frac{1}{t^2}u_{xx}\right) \\ &= \frac{1}{t^{5/2}}e^{x^2/2t}(x^2u + 2xu_x + tu + u_{xx}). \end{aligned}$$

- It follows that $v_t = -\frac{1}{2}v_{xx}$.